Classification of D-finite walks in the quarter plane via elliptic functions

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Joint work with Thomas Dreyfus and Kilian Raschel

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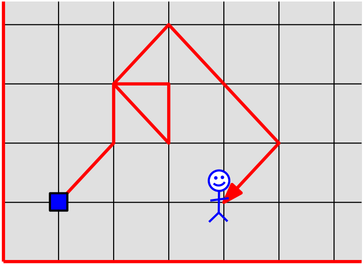
Introduction: Walks in 1/4-plane



QUADRANT WALKS

Model: Fix *step-probabilities* $p_{i,j}$ for $i, j \in \{-1, 0, 1\}$ with $\sum_{i,j} p_{i,j} = 1$.

Walk definition:

- $(x_0, y_0) = (0, 0)$
- For each k, independently $(x_{k+1}, y_{k+1}) = (x_k + i, y_k + j)$ with probability $p_{i,j}$

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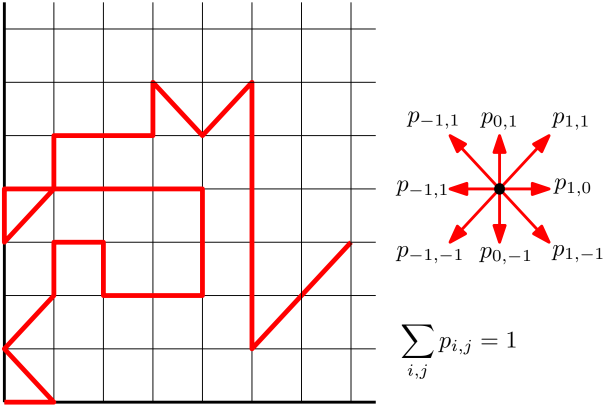
Aim: Determine probability $q_{n,a,b}$ that $(x_n, y_n) = (a, b)$ and $x_k, y_k \geq 0$ for all $k \leq n$.

Equivalent aim: determine the generating function

$$\mathsf{Q}(t, x, y) := \sum_{n,a,b=0}^{\infty} q_{n,a,b} t^n x^a y^b.$$

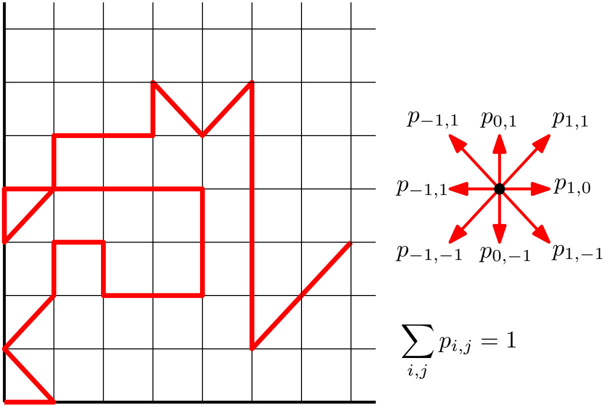
RANDOM WALKS IN THE QUADRANT



Let $q_{n,a,b}$ be the probability that a random walk of length n stays in the quarter plane and ends at (a, b). i.e.,

$$q_{n,a,b} = \sum_{\substack{\text{length } n \text{ paths} \\ (0,0)\to(a,b)}} \left(\prod_{steps} p_{i,j} \right)$$

Quadrant walks: generating function



$$\sum_{i,j} p_{i,j} = 1$$

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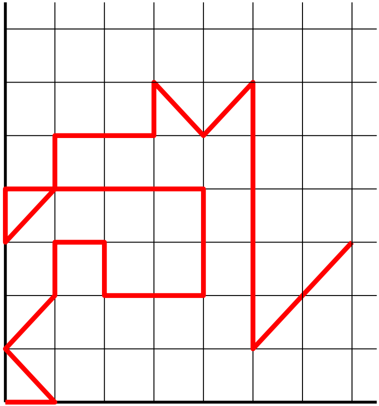
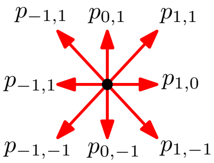
QUADRANT WALKS: GENERATING FUNCTION





$$\sum_{i,j} p_{i,j} = 1$$

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Aim: Classify $\mathsf{Q}(x,y;t)$ into hierarchy Rational/Algebraic/etc.

Classifying generating series

For a series (or a function) $F(t)$, the following properties satisfy

$$\text{Rational} \Rightarrow \text{Algebraic} \Rightarrow \text{D-finite} \Rightarrow \text{D-Algebraic}:$$

- **Rational:** $F(t) = \frac{P(t)}{Q(t)}$ for polynomials $P(t)$ and $Q(t)$.
- **Algebraic:** $P(F(t), t) = 0$ for some non-zero polynomial $P(x)$.
- **D-finite:** $F(t)$ satisfies some non-trivial linear differential equation. E.g.

$$t^3 F''(t) + t^2 F'(t) + (t+1)F(t) - 1 = 0$$

- **D-algebraic:** $F(t)$ satisfies some non-trivial algebraic differential equation. E.g.

$$t^2 F'(t) + F''(t)F(t) + tF(t) = 0$$

For multivariate functions/series: Classification considered separately with respect to each variable

MAIN RESULTS

Theorem 1: the following are equivalent:

- The group* of the model is finite
- $Q(t,x,y)$ satisfies a linear DE in t with coefficients polynomial in t,x,y
- $Q(t,x,y)$ satisfies a linear DE in x with coefficients polynomial in t,x,y
- $Q(t,x,y)$ satisfies a linear DE in y with coefficients polynomial in t,x,y

* Group to be defined later.

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Theorem 2: the following are equivalent:

- The group* of the model is finite and the orbit sum* of the model is 0
- There is some non-zero polynomial $P(q,t,x,y)$ satisfying $P(Q(t,x,y),t,x,y)=0$. (that is $Q(t,x,y)$ is algebraic)

* Group and orbit sum to be defined later.

GROUP OF THE WALK AND ORBIT SUM

Group definition: Consider the rational transformations

$$\nu_1(x,y) = \left(x, \frac{p_{-1,-1}y^{-1} + p_{-1,0} + p_{-1,1}y}{(p_{1,-1}y^{-1} + p_{1,0} + p_{1,1}y)x}\right)$$
$$\nu_2(x,y) = \left(\frac{p_{-1,-1}x^{-1} + p_{0,-1} + p_{1,-1}x}{(p_{-1,1}x^{-1} + p_{0,1} + p_{1,1}x)y}, y\right),$$

which both fix

$$P(x,y) = \sum_{i,j\in\{-1,0,1\}} p_{i,j}x^i y^j.$$

The *group of the walk* is the group G generated by ν_1 and ν_2.
Orbit sum definition: If the group G is finite, the orbit sum $O(x,y)$ is

$$O(x,y) = \sum_{g\in G}(-1)^{|g|} g\cdot(xy).$$

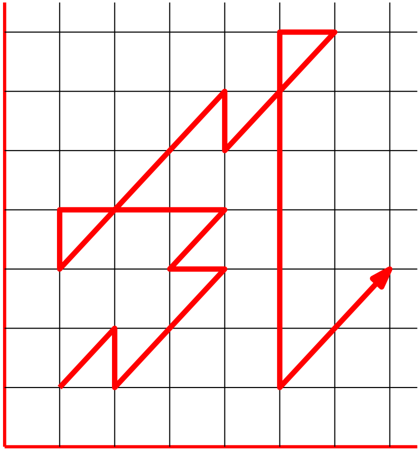
Background: Unweighted quadrant walks

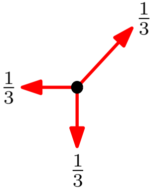
Unweighted problem: choose step set $S \in \{-1,0,1\}^2 \setminus \{(0,0)\}$ then set $p_{i,j} = \frac{1}{|S|}$ for $(i,j) \in S$ and $p_{i,j} = 0$ for $(i,j) \notin S$.
Enumerative result: $|S|^n q_{i,j,n}$ is the number of paths in the 1/4-plane with steps in S from $(0,0)$ to (i,j) of length n.
Equivalently: Determine the generating function

$$\mathsf{Q}(x,y;t) := \sum_{n\geq 0} \sum_{i,j\geq 1} q_{i,j,n} t^n x^i y^j.$$

Systematic approach: 79 distinct non-trivial step sets identified [Bousquet-Mélou, Mishna, 2010].

Example: Kreweras paths





$$\mathsf{Q}(t,x,y) := \sum_{a,b=0}^{\infty} \sum_{\substack{\text{paths from} \\ (0,0) \text{ to } (a,b)}} \left(\frac{t}{3}\right)^{\#\text{steps}} x^a y^b.$$

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Problem: $|S|^n q_{i,j,n}$ is the number of paths in the 1/4-plane with steps in S from $(0,0)$ to (i,j) of length n.

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All models now classified using many methods

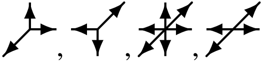
- Algebraic methods [Malyshev, Bousquet-Mélou, Mishna]
- Asymptotic analyses [Denisov, Wachtel, Mishna, Rechnitzer]
- Computer algebra [Bostan, Chyzak, Van Hoeij, Kauers, Pech]
- Galois Theory [Dreyfus, Hardouin, Roques, Singer]
- Analytic approach [Fayolle, Raschel, Kurkova, Bernardi]

Unweighted quadrant walks

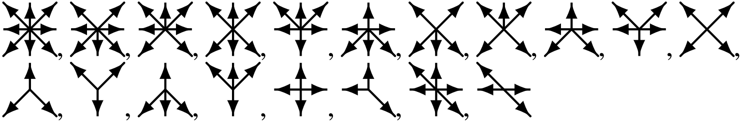
In total: 79 different non-trivial step sets S.

Generating function $\mathsf{Q}(x, y; t)$ is...

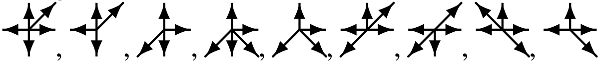
- Algebraic in 4 cases:



- D-finite in 19 further cases:



- D-algebraic in 9 further cases:



- In remaining 47 cases, $Q(t, x, y)$ is not D-algebraic.

[Bousquet-Mélou, Mishna, Denisov, Wachtel, Rechnitzer, Bostan, Chyzak, Van Hoeij, Kauers, Pech, Dreyfus, Hardouin, Roques, Singer, Fayolle, Raschel, Kurkova, Bernardi]

UNWEIGHTED QUADRANT WALKS

In total: 79 different non-trivial step sets S.

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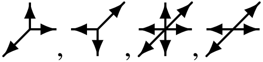

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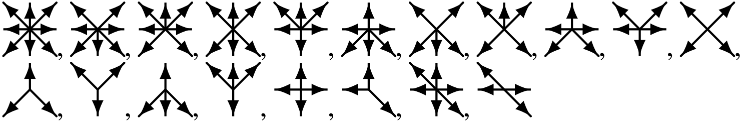

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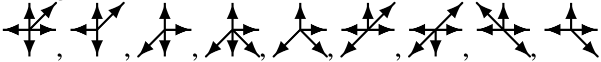

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This work: Generalisation to weighted walks, single systematic method

BACKGROUND: WEIGHTED QUADRANT WALKS

Previous parts of classification:

- If group infinite: decoupling function $\iff$ $\mathsf{Q}(x, y, t)$ D-algebraic [Dreyfus, Hardouin, 2021],[Hardouin, Singer 2021],[Dreyfus, 2023]
- For fixed t: Infinite group $\Rightarrow$ $\mathsf{Q}(x, y, t)$ not D-finite in x, y [Kurkova, Raschel, 12],[E.P., 24]
- For fixed t: Finite group $\Rightarrow$ $\mathsf{Q}(x, y, t)$ D-finite in x, y [Fayolle, Raschel, 10],[Dreyfus, Raschel, 20]
- For fixed t and group finite: orbit sum 0 $\iff$ $\mathsf{Q}(x, y, t)$ algebraic in x, y[Dreyfus, Raschel, 20]

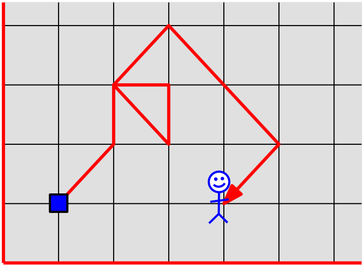
Other results:

- The group is infinite or has order at most 12 [Jiang, Tanakoli, Zhao, 2021][Hardouin, Singer, 2021]
- All models with groups of order ≤ 8 explicitly classified [Kauers, Yatchak, 2015]

TALK OUTLINE

- **Part 1:** Walks in 1/4-plane functional equations
- **Part 2:** Classification of $\mathsf{Q}(x, y, t)$ in x, y for fixed t
- **Part 3:** Full classification of $\mathsf{Q}(x, y, t)$

Part 1a: Walks in 1/4-plane functional equation

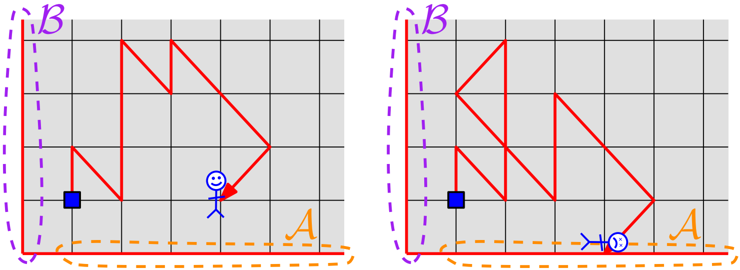


Concept: The walker dies when they touch an axis.
Quadrant walk: Walk from $(1,1)$ not touching an axis.
Dead walk: Walk from $(1,1)$ only touching an axis at the end.

(**Quadrant walk+step**) or **empty walk** $=$ **Quadrant walk** or **Dead walk**

Generating function $D(x,y;t)$ for dead walks splits as

$$D(x,y;t) = A(x;t) + B(y;t).$$



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$$Q(x, y; t)tP(x, y) + xy = Q(x, y; t) + D(x, y; t).$$

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$$Q(x,y;t)tP(x,y) + xy = Q(x,y;t) + D(x,y;t).$$

To simplify, write Kernel $K(x,y;t) = 1 - tP(x,y)$.
To solve:

$$xy = K(x,y;t)Q(x,y;t) + A(x;t) + B(y;t).$$

Unknowns: $Q(x,y;t), A(x;t), B(y;t)$.

Part 1b: Algebraic functional equation → analytic functional equation

Origin: [Fayolle, Iasnogorodski, 1979], [Fayolle, Ianogorodski, Malyshev, 1999] and [Raschel, 2010]

Also used in: [Fayolle, Raschel, 2010], [Kurkova, Raschel, 2012], [Dreyfus, Raschel, 2019], [Bernardi, Bousquet-Mélou, Raschel, 2021], [Dreyfus, Hardouin, Roques, Singer, 2018], [Dreyfus, Hardouin, Roques, Singer, 2020], [Hardouin, Singer, 2021], etc.

QUADRANT WALKS SOLUTION

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Solution idea:

Step 1: Fix $t \in (0, 1)$ and define $\mathcal{K} = \{(x, y) : K(x, y; t) = 0\}$. Then for $(x, y) \in \mathcal{K}$, and $|x|, |y| < 1$, we have

$$A(x) + B(y) = xy.$$

Classic theorem: $\mathcal{K}$ is a surface with genus 1 (usually), so is homeomorphic to some $\mathbb{C}/(\omega_1\mathbb{Z} + \omega_2\mathbb{Z})$, where $\omega_1 \in i\mathbb{R}$ and $\omega_2 \in \mathbb{R}$.

Quadrant walks solution

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Step 2: Parametrise $\mathcal{K}$ using elliptic functions $X, Y : \mathbb{C} \to \mathbb{C} \cup \{\infty\}$, satisfying

- $X(\omega) = X(\omega + \omega_1) = X(\omega + \omega_2)$
- $Y(\omega) = Y(\omega + \omega_1) = Y(\omega + \omega_2)$

To solve: (for A and B)

$$A(X(\omega)) + B(Y(\omega)) = X(\omega)Y(\omega) \qquad \text{when } |X(\omega)|, |Y(\omega)| < 1.$$

ASIDE: ELLIPTIC FUNCTIONS

Definition: An *elliptic* function f is a meromorphic function $f : \mathbb{C} \to \mathbb{C} \cup \{\infty\}$ with two independent periods $\omega_1, \omega_2 \in \mathbb{C}$, that is

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Definition: The Weierstrass function $\wp$ with periods ω_1 and ω_2 is defined by

$$\wp(\omega) := \frac{1}{\omega^2} + \sum_{\ell \in (\omega_1\mathbb{Z} + \omega_2\mathbb{Z}) \setminus 0} \left(\frac{1}{\left(\omega + \ell\right)^2} - \frac{1}{\ell^2} \right).$$

This has a double pole at each point in $\omega_1\mathbb{Z} + \omega_2\mathbb{Z}$ and no other poles

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This has a double pole at each point in $\omega_1\mathbb{Z} + \omega_2\mathbb{Z}$ and no other poles
Theorem (Liouville): The only *holomorphic* elliptic functions are constant functions.
Corollary 1: Any two elliptic functions f and g with the same periods are algebraically related.
Corollary 2: Any elliptic function with periods ω_1 and ω_2 is a rational function of $\wp(\omega, \omega_1, \omega_2)$ and $\wp'(\omega, \omega_1, \omega_2)$

More about $X(z)$ and $Y(z)$

$X(\omega)$ and $Y(\omega)$ can be written as

$$X(\omega) = x_0 + \frac{x_1}{\wp(\omega) + x_2},$$
$$Y(\omega) = y_0 + \frac{y_1}{\wp(\omega - \omega_3/2) + y_2},$$

for explicit values periods ω_1, ω_2 and values $\omega_3, x_0, x_1, x_2, y_0, y_1, y_2$ determined by the step set and t.

Weierstrass function:

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satisfies $\wp(\omega) = \wp(\omega + \omega_1) = \wp(\omega + \omega_2) = \wp(-\omega)$

MORE ABOUT $X(z)$ AND $Y(z)$

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Inherited transformation properties: $X(\omega)$ and $Y(\omega)$ satisfy

- $X(\omega) = X(\omega+\omega_1) = X(\omega+\omega_2) = X(-\omega)$
- $Y(\omega) = Y(\omega+\omega_1) = Y(\omega+\omega_2) = Y(\omega_3-\omega)$

QUADRANT WALKS SOLUTION

To solve: (for A and B)

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Step 3: Define

$$\tilde{A}(\omega) = A(X(\omega)), \qquad \text{for } \Re(\omega) \in \left(-\frac{\omega_3}{2}, \frac{\omega_3}{2}\right),$$

$$\tilde{B}(\omega) = B(Y(\omega)), \qquad \text{for } \Re(\omega) \in (0, \omega_3).$$

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To solve:

$$\tilde{A}(\omega) + \tilde{B}(\omega) = X(\omega)Y(\omega),$$

given that

$$\tilde{A}(\omega) = \tilde{A}(\omega + \omega_1) = \tilde{A}(-\omega),$$
$$\tilde{B}(\omega) = \tilde{B}(\omega + \omega_1) = \tilde{B}(\omega_3 - \omega).$$

$\tilde{A}$ and $\tilde{B}$ are holomorphic
$\tilde{A}$ and $\tilde{B}$ extend to meromorphic functions on $\mathbb{C}$
$\tilde{A}$ has roots at roots of $X(\omega)$

Part 2: Analytic functional equation $\rightarrow$ nature in x.

[Fayolle, Raschel, 10],[Dreyfus, Raschel, 20]

GROUP OF THE WALK

Recall: Group generated by transformations $\nu_1(x, y)$ which fixes x and $P(x, y)$ and $\nu_2(x, y)$ which fixes y and $P(x, y)$.

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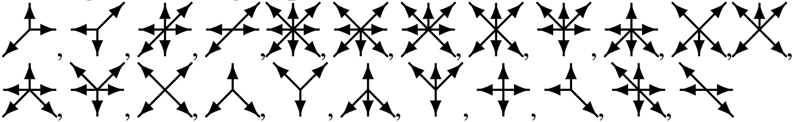

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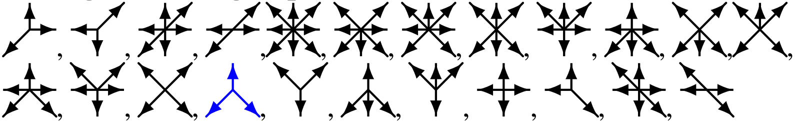

Example: in this case $P(x, y) = \frac{1}{3}\left(\frac{1}{xy} + \frac{x}{y} + y\right)$, so

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Finite group: $\left\{(x, y), \left(\frac{1}{x}, y\right), \left(\frac{1}{x}, \frac{\frac{1}{x} + x}{y}\right), \left(x, \frac{\frac{1}{x} + x}{y}\right)\right\}$

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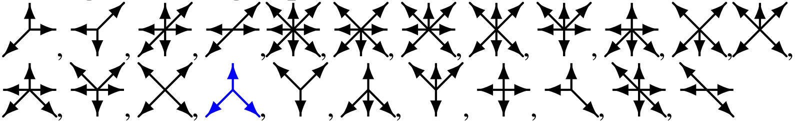

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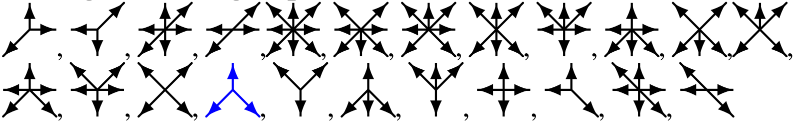

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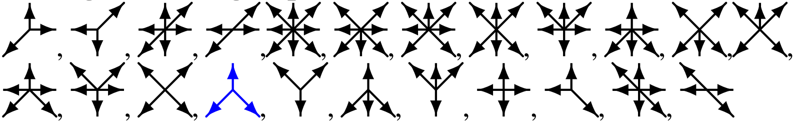

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$$O(x,y) = xy - \frac{1}{x}y + \frac{1}{x}\left(\frac{\frac{1}{x} + x}{y}\right) - x\frac{\frac{1}{x} + x}{y} = \frac{(x^4 - 1)(xy^2 - x^2 - 1)}{x^2 y} \neq 0$$

ORBIT SUM

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$$O(x,y)=\sum_{g\in G}(-1)^{|g|}g\cdot(xy).$$

Example: in the case $P(x,y)=\frac{1}{9}\left(\frac{1}{x}+\frac{y}{x}+2y+xy+2x+\frac{x}{y}+\frac{1}{y}\right)$,

$$\nu_1\cdot f(x,y)=f\left(x,\frac{x}{y(1+x)}\right)\quad\text{and}\quad\nu_2\cdot f(x,y)=f\left(\frac{y}{x(1+y)},y\right).$$

Finite group:

$$\left\{(x,y),\left(\frac{y}{x(1+y)},y\right),\left(\frac{y}{x(1+y)},\frac{1}{x+y+xy}\right),\left(\frac{x}{y(1+x)},\frac{1}{x+y+xy}\right),\left(\frac{x}{y(1+x)},x\right),\right.$$
$$\left.(y,x),\left(y,\frac{y}{x(1+y)}\right),\left(\frac{1}{x+y+xy},\frac{y}{x(1+y)}\right),\left(\frac{1}{x+y+xy},\frac{x}{y(1+x)}\right),\left(x,\frac{x}{y(1+x)}\right)\right\}$$

Orbit sum:

$$O(x,y)=xy-\frac{y^2}{x(y+1)}+\cdots-\frac{x^2}{y(x+1)}=0.$$

ORBIT SUM 0 → ALGEBRAIC IN x, y (FOR FIXED t)

To solve:

$$\tilde{A}(\omega) + \tilde{B}(\omega) = X(\omega)Y(\omega),$$

given that

$$\tilde{A}(\omega) = \tilde{A}(\omega + \omega_1) = \tilde{A}(-\omega),$$
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Solution: Combining the equations yields

$$\tilde{B}(\omega+\omega_3)-\tilde{B}(\omega)=\tilde{B}(-\omega)-\tilde{B}(\omega)=X(-\omega)Y(-\omega)-X(\omega)Y(\omega)$$

Orbit sum 0 → algebraic in x, y (for fixed t)

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If orbit sum $O(x, y) = 0$, then $N\omega_3$ and ω_1 are periods of both $\tilde{B}(\omega)$ and $Y(\omega)$, so $P(Y(\omega), \tilde{B}(\omega)) = 0$ for some polynomial P.

ORBIT SUM $0 \to$ ALGEBRAIC IN x, y (FOR FIXED t)

From last slide: If group finite and orbit sum $O(x,y) = 0$, there is a polynomial P satisfying

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Therefore

$$P(y, B(y)) = 0,$$

so B is algebraic. Similarly $A(x)$ is algebraic, so the generating function

$$\mathsf{Q}(x, y) = \frac{xy - A(x) - B(y)}{K(x, y)}$$

is algebraic.

FINITE GROUP → D-FINITE IN x (FOR FIXED t)

If the group is finite but orbit sum is not 0:

Recall:

$$\tilde{B}(\omega + N\omega_3) - \tilde{B}(\omega) = O(X(\omega), Y(\omega)).$$

Then $G(\omega) := \frac{\tilde{B}(\omega)}{O(X(\omega),Y(\omega))}$ satisfies

$$G(\omega + M\omega_2) - G(\omega) = 1 \quad \Longrightarrow \quad G'(\omega + M\omega_2) - G'(\omega) = 0,$$

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so $G'(\omega)$ algebraic in $Y(\omega)$. Writing $G'(\omega)$ in terms of $B'(Y(\omega))$ and $B(Y(\omega))$ yields an equation.

$$a_1(y)B'(y) + a_2(y)B(y) + a_3(y) = 0,$$

where a_1, a_2, a_3 are algebraic. It follows that B is D-finite.

Part 3: Analytic functional equation $\rightarrow$ nature in t.

Recall analytic characterisation: For fixed t, $X(\omega)$ and $Y(\omega)$ explicit elliptic functions (depending on model and t).

$\tilde{A}$ and $\tilde{B}$ determined by $\tilde{A}(\omega)+\tilde{B}(\omega)=X(\omega)Y(\omega)$ and

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along with information about poles.

$A(x)$ and $B(y)$ determined by
$\tilde{A}(\omega)=A(X(\omega))$ and $\tilde{B}(\omega)=B(Y(\omega))$ for $\Re(\omega)\in(0,\omega_3/2)$

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For classification in t: consider all parameters as function of t.

CLASSIFICATION (WITH t VARIABLE INCLUDED)

X and Y are given by

$$X(\omega,t)=a(t)+\frac{D_1'(a(t),t)}{\wp(\omega,\omega_1(t),\omega_2(t))-\frac{1}{6}D_1''(a(t),t)}$$

$$Y(\omega,t)=b(t)+\frac{D_2'(b(t),t)}{\wp(\omega-\omega_3(t)/2,\omega_1(t),\omega_2(t))-\frac{1}{6}D_2''(b(t),t)},$$

Where D_1 and D_2 are polynomials and a,b are algebraic in t

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Where D_1 and D_2 are polynomials and a, b are algebraic in t

$\tilde{A}$ and $\tilde{B}$ determined by $\tilde{A}(\omega,t) + \tilde{B}(\omega,t) = X(\omega,t)Y(\omega,t)$ and

$$\tilde{A}(\omega,t) = \tilde{A}(\omega+\omega_1(t),t) = \tilde{A}(-\omega,t),$$

$$\tilde{B}(\omega,t) = \tilde{B}(\omega+\omega_1(t)) = \tilde{B}(\omega_3(t)-\omega),$$

along with information about poles.

Finite group, orbit sum 0 **case:** $\tilde{A}$ can be written explicitly in terms of $\wp(\omega,\omega_1(t),N\omega_2(t))$, $\omega_3(t)$ and poles and residues of $X(\omega,t)Y(\omega,t)$.

ORBIT SUM 0 $\to$ ALGEBRAIC

Claim: Finite group and orbit sum $= 0$ implies $\mathsf{Q}(x, y, t)$ is algebraic.
Proof outline: Write $\wp_t(\omega) := \wp(\omega, \omega_1(t), \omega_2(t))$. We prove that each function lies in one of three sets:

- $\overline{\mathbb{C}(t)}$: algebraic functions of t
- $\overline{\mathbb{C}(t, \wp_t(\omega))}$: algebraic functions of t and $\wp_t(\omega)$
- $U_t = \{a(t) : \wp_t(a(t)) \in \overline{\mathbb{C}(t)}\}$

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Steps in solution:

Claim 1: $\omega_1(t), \omega_2(t), \omega_3(t) \in U_t$

Claim 2: $X(\omega, t), Y(\omega, t), \wp_t'(\omega) \in \overline{\mathbb{C}(t, \wp_t(\omega))}$

Claim 3: Each pole $\omega = \alpha(t)$ of $X(\omega, t)$ or $Y(\omega, t)$ lies in U_t

Claim 4: Each pole of $\tilde{A}(\omega, t)$ lies in U_t

Claim 5: Each residue of $X(\omega, t)$ or $Y(\omega, t)$ lies in $\overline{\mathbb{C}(t)}$

Claim 6: Each residue of $\tilde{A}(\omega, t)$ lies in $\overline{\mathbb{C}(t)}$

Claim 7: $\wp(\omega, \omega_1(t), N\omega_2(t)) \in \overline{\mathbb{C}(t, \wp_t(\omega))}$

Claim 8: $\tilde{A}(\omega, t) \in \overline{\mathbb{C}(t, \wp_t(\omega))}$.

Algebraicity of parametrisation

For first step we use a different characterisation of $\wp(\omega)$:
Definition: The Weierstrass function $\wp$ with *invariants* g_2 and g_3 is the unique solution to the differential equation

$$\wp'(\omega)^2 = 4\wp(\omega)^3 - g_2\wp(\omega) - g_3$$

satisfying $\wp(\omega) \sim \omega^{-2}$ as $\omega \to 0$.

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Lemma: the invariants $g_2(t)$ and $g_3(t)$ of $\wp_t$ are explicit algebraic functions of t.
Proof: Writing $Y(\omega)$ and $X(\omega)$ in terms of $\wp_t(\omega)$ and $\wp_t'(\omega)$, the equation

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Corollary (Claim 2): $X(\omega, t), Y(\omega, t), \wp_t'(\omega) \in \overline{\mathbb{C}(t, \wp_t(\omega))}$

ORBIT SUM 0: PROOF OF ALGEBRAICITY

Recall:

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- $\overline{\mathbb{C}(t, \wp_t(\omega))}$: algebraic functions of t and $\wp_t(\omega)$
- $U_t = \{a(t) : \wp_t(a(t)) \in \overline{\mathbb{C}(t)}\}$

Orbit sum 0: Proof of algebraicity

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From explicit parametrisation: Each pole $\omega = \alpha(t)$ of $X(\omega, t)$ or $Y(\omega, t)$ lies in U_t and each residue of $X(\omega, t)$ or $Y(\omega, t)$ lies in $\overline{\mathbb{C}(t)}$.

To $\tilde{B}$: Recall, if $\frac{\omega_3}{\omega_2} = \frac{M}{N}$ then

$$\tilde{B}(N\omega_3 + \omega) - \tilde{B}(\omega) = O(X(\omega), Y(\omega)) = 0.$$

$\rightarrow \tilde{B}$ can be written explicitly in terms of $\wp(\omega, \omega_1, M\omega_2) \in \overline{\mathbb{C}(t, \wp_t(\omega))}$ and its poles and residues.

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$\rightarrow \tilde{B}(\omega) \in \overline{\mathbb{C}(t, \wp_t(\omega))}$, so $\tilde{B}(\omega)$ is algebraic in t and $Y(\omega)$

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To $\tilde{B}$: Recall, if $\frac{\omega_3}{\omega_2} = \frac{M}{N}$ then

$$\tilde{B}(N\omega_3 + \omega) - \tilde{B}(\omega) = O(X(\omega), Y(\omega)) = 0.$$

$\rightarrow \tilde{B}$ can be written explicitly in terms of $\wp(\omega, \omega_1, M\omega_2) \in \overline{\mathbb{C}(t, \wp_t(\omega))}$ and its poles and residues.

These poles and residues can be written in terms of those of $X(\omega)Y(\omega)$, so they also lie in U_t (for poles) and $\overline{\mathbb{C}(t)}$ (for residues).

$\rightarrow \tilde{B}(\omega) \in \overline{\mathbb{C}(t, \wp_t(\omega))}$, so $\tilde{B}(\omega)$ is algebraic in t and $Y(\omega)$

$\rightarrow B(y)$ algebraic, similarly $A(x)$ algebraic, so $\mathsf{Q}(x, y, t)$ is algebraic.