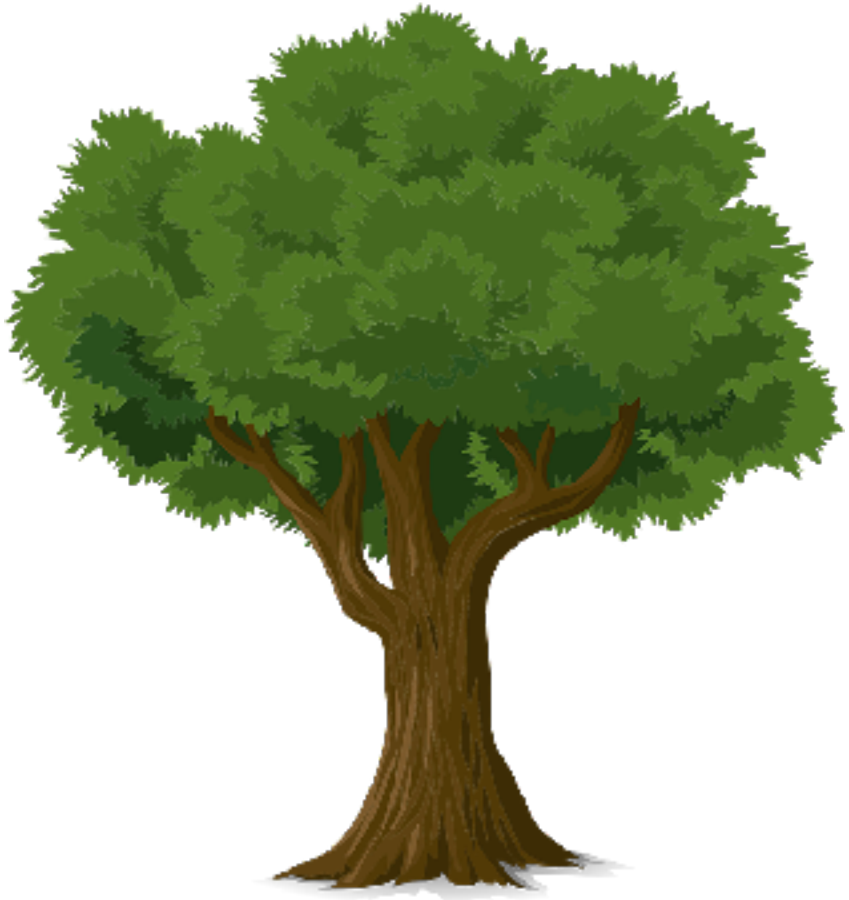
Parking on Cayley trees

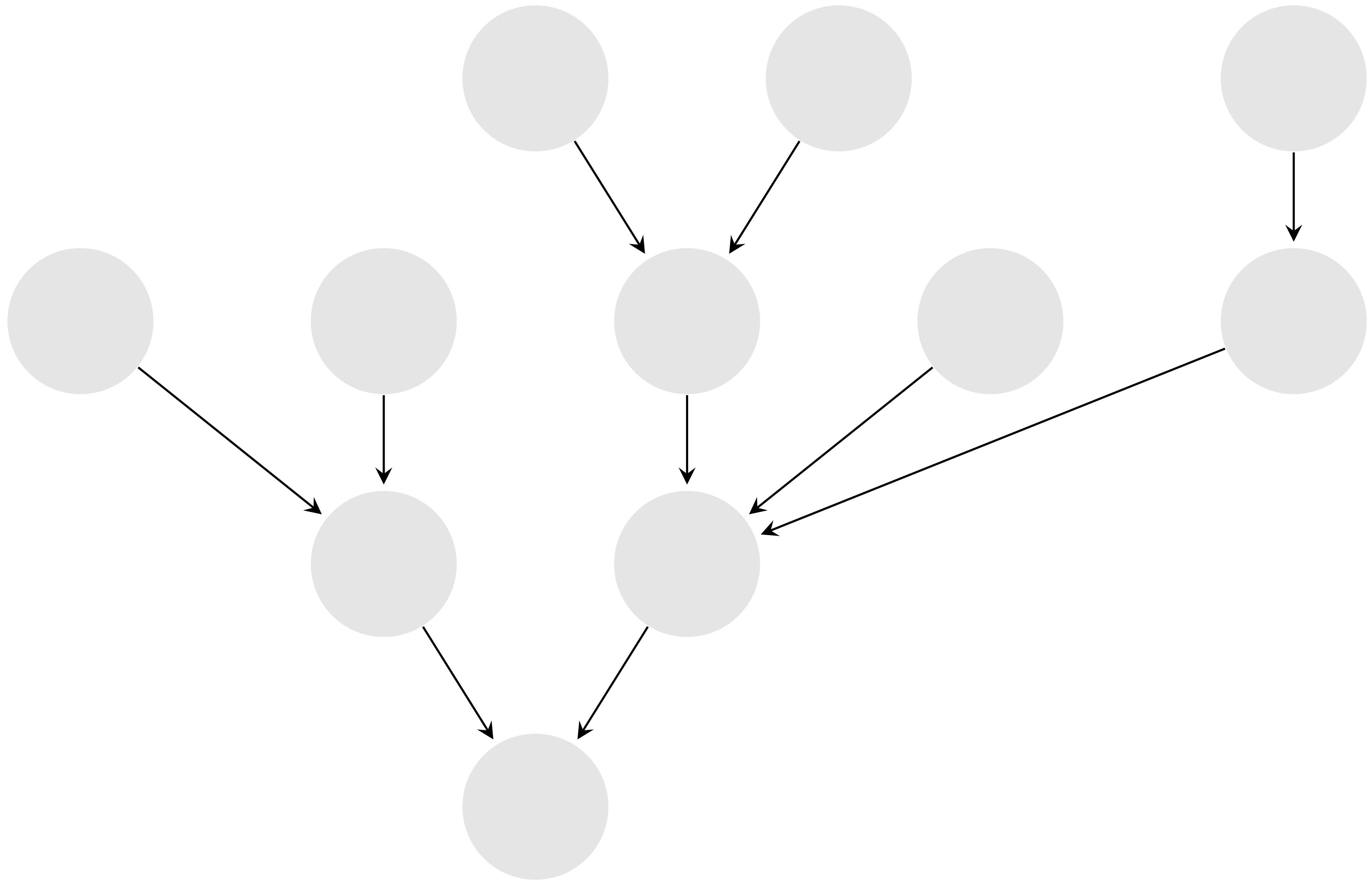


Journées Math-STIC 2024

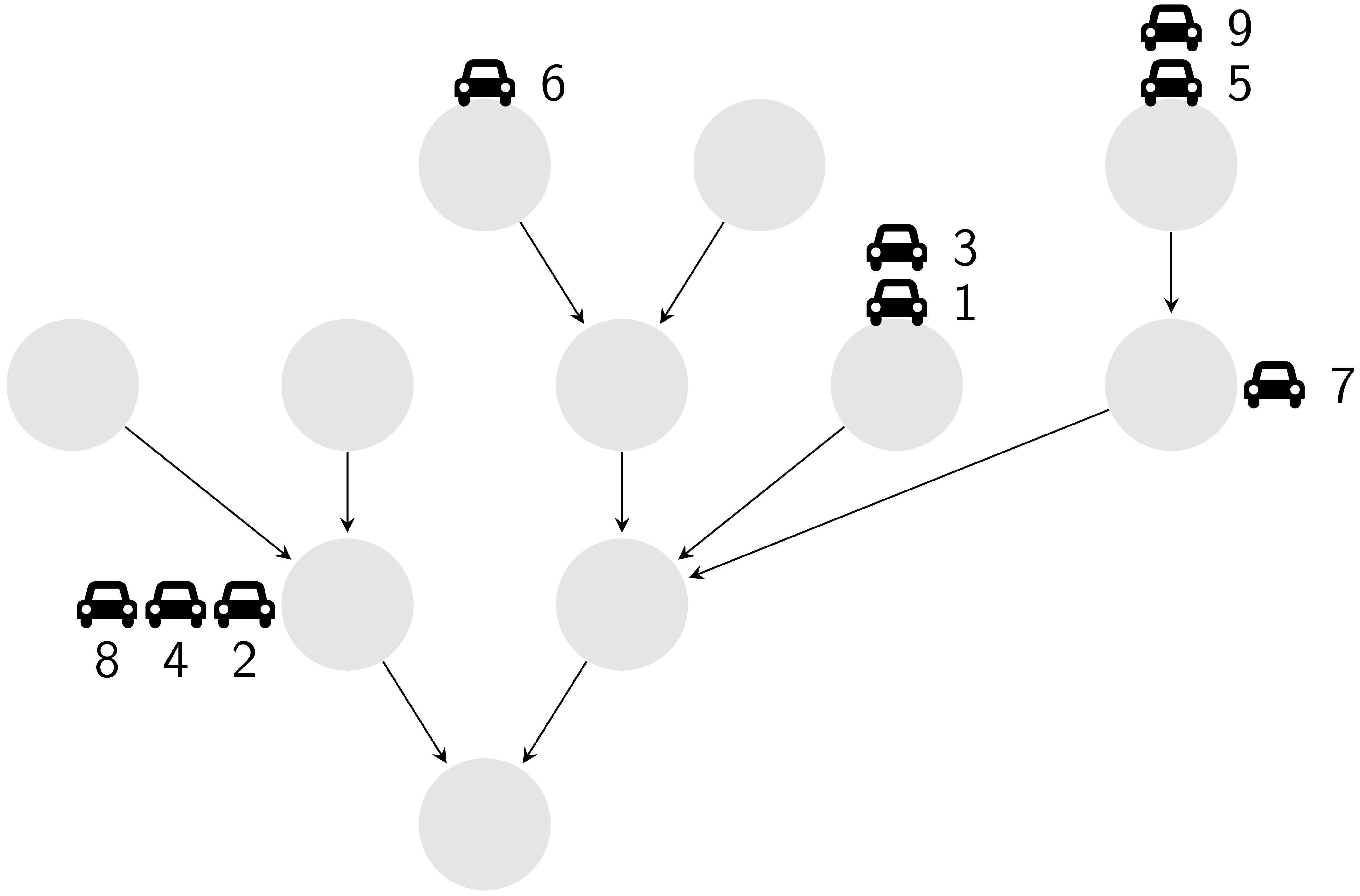
4th October 2024

Alice CONTAT

Parking rules on a tree



Parking rules on a tree



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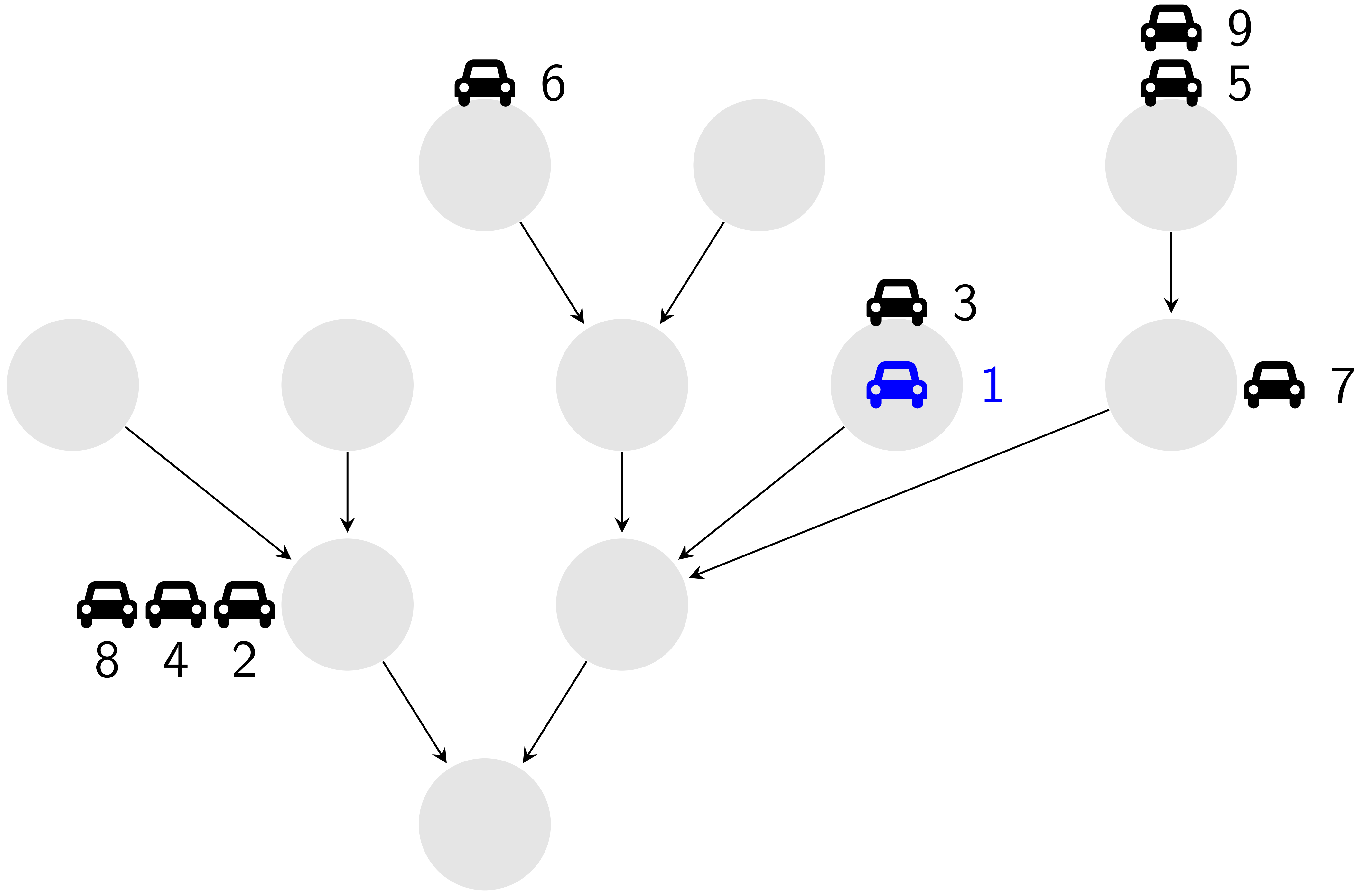

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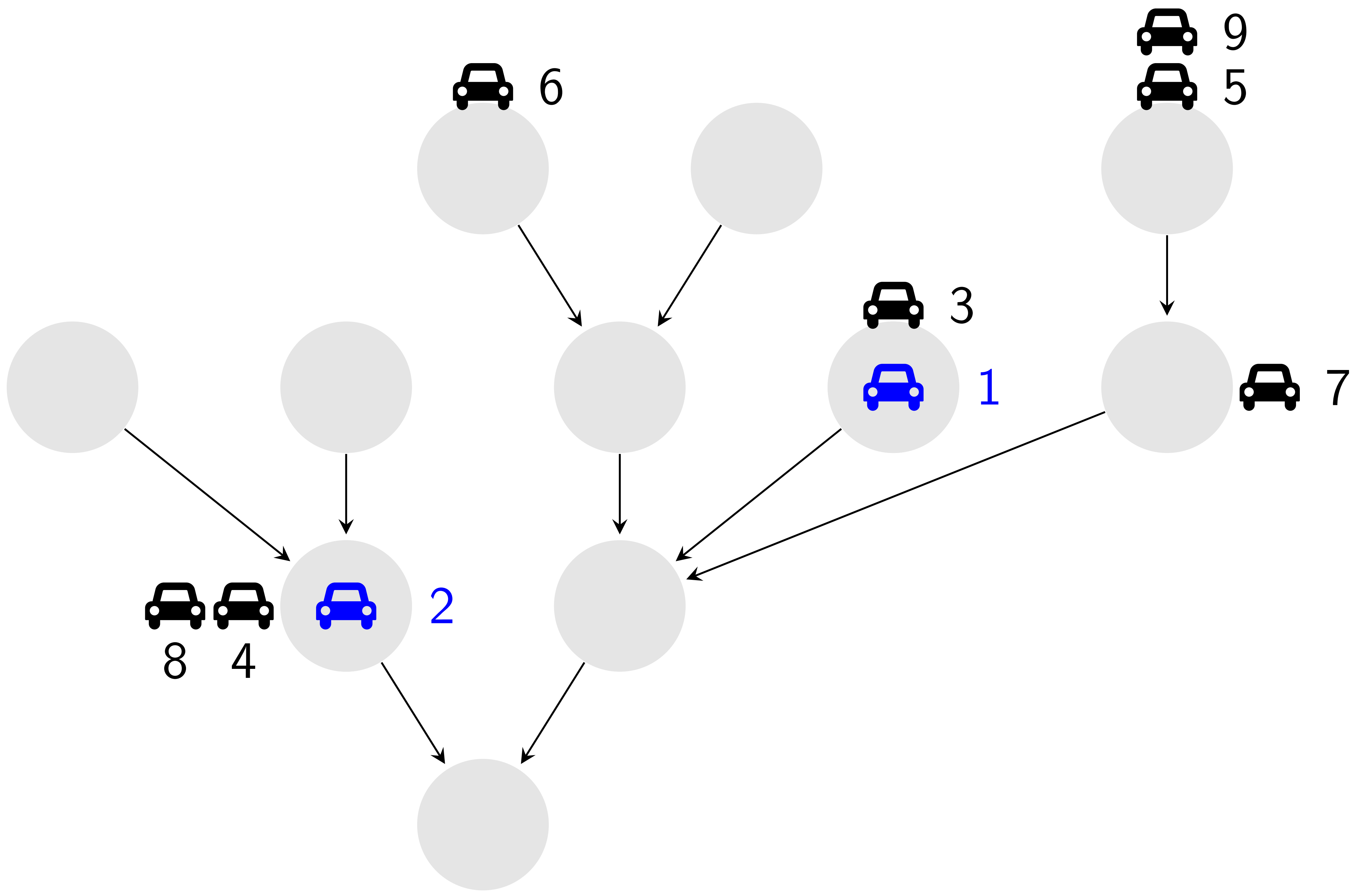

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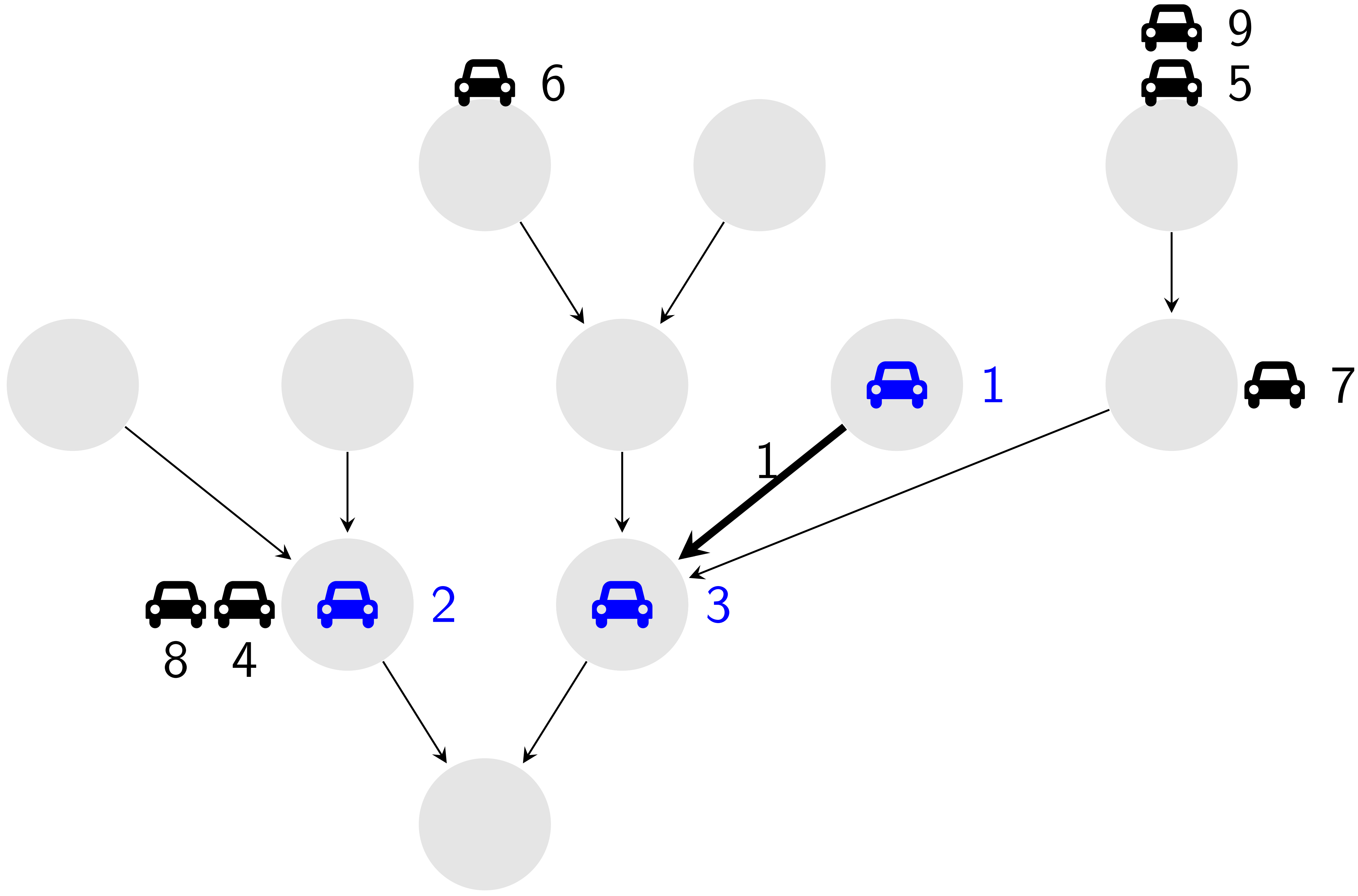

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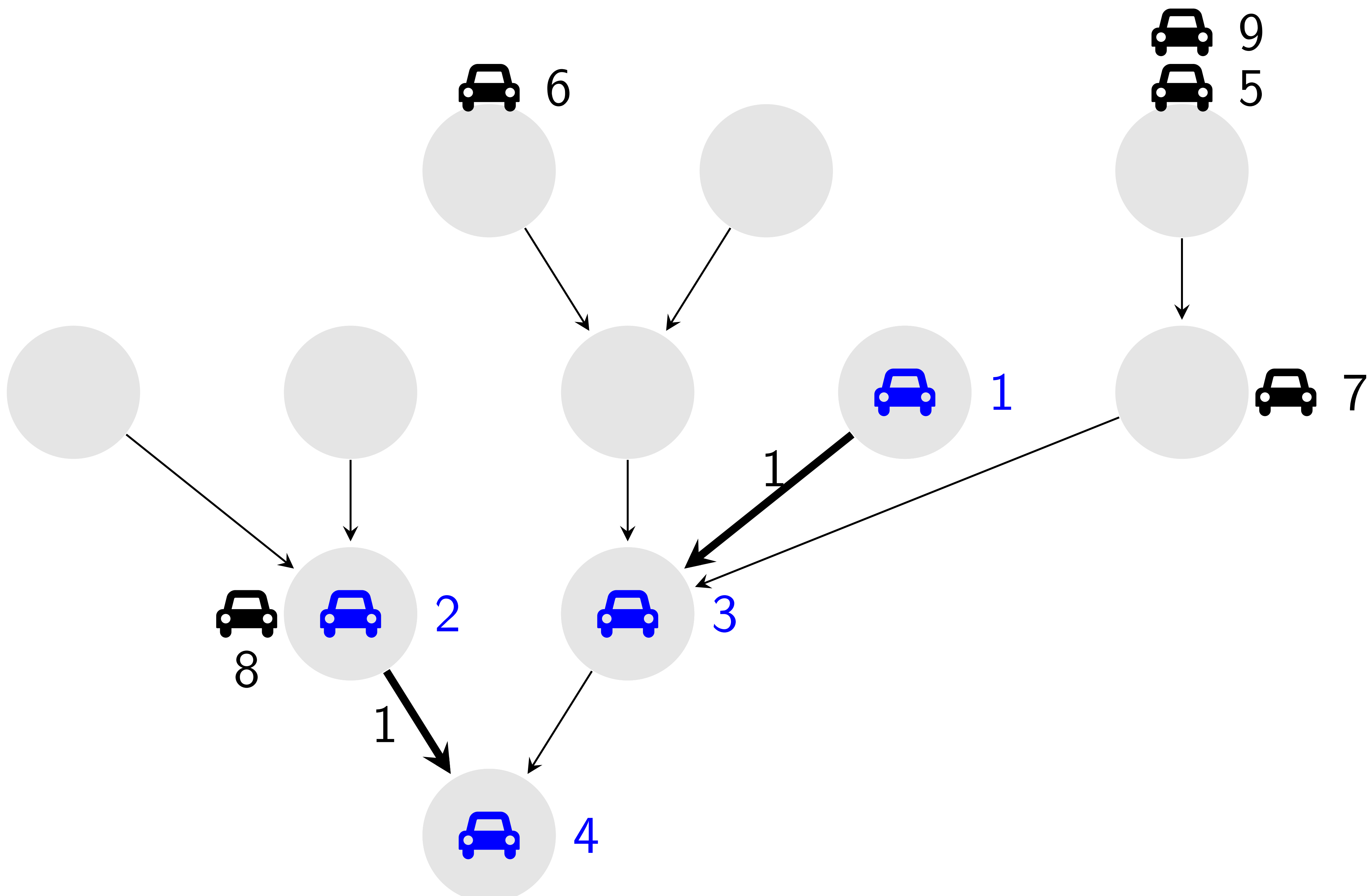

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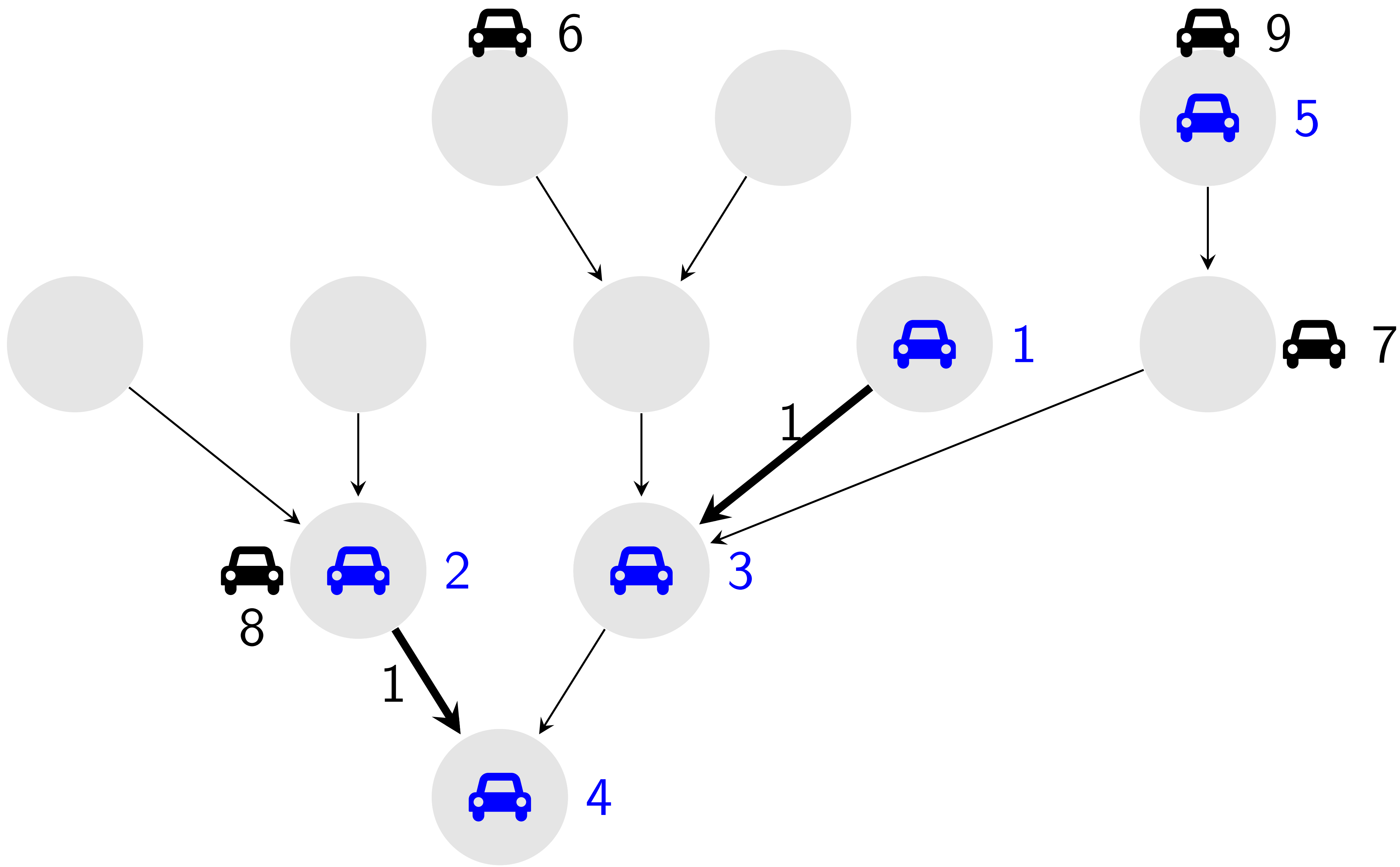

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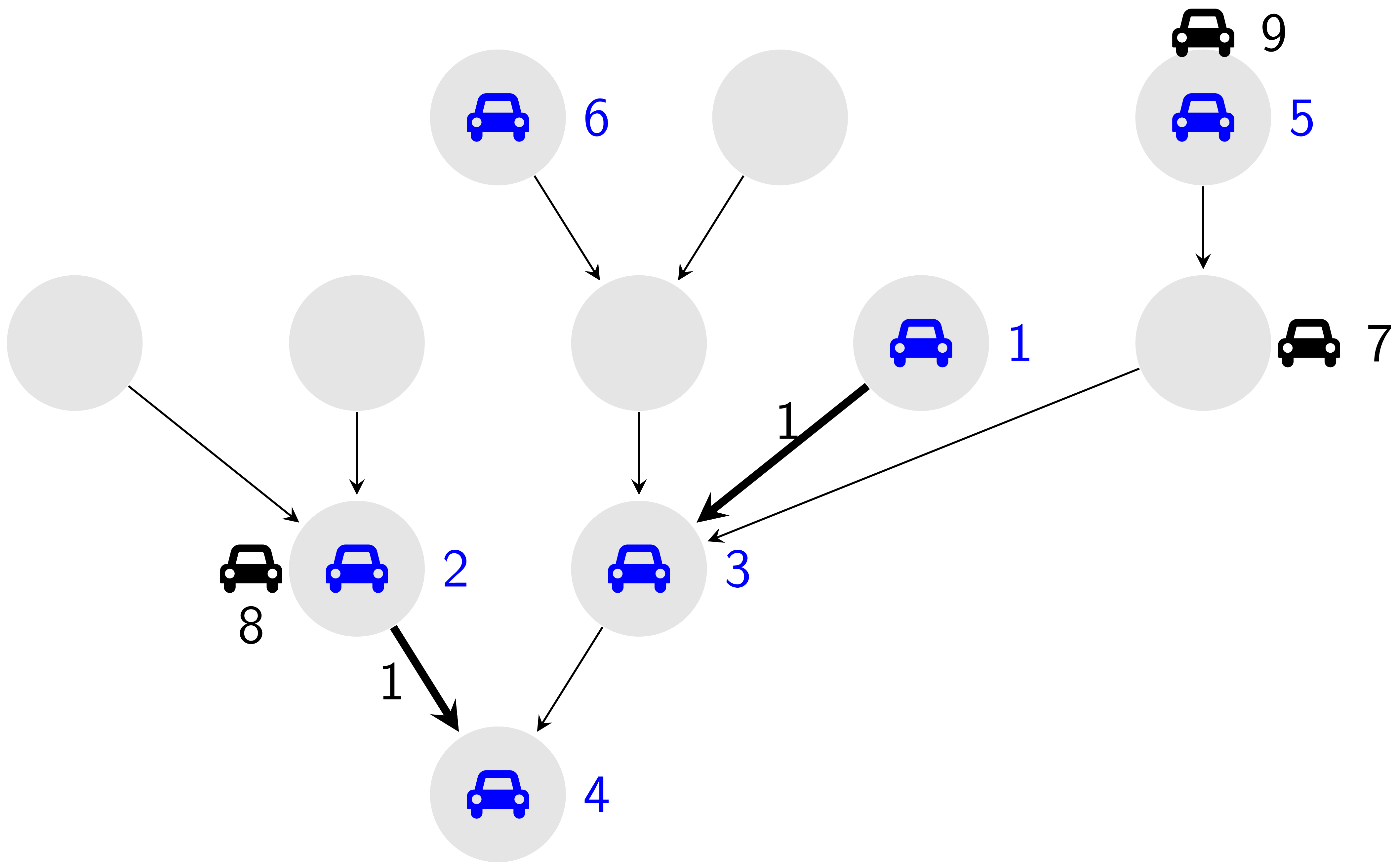

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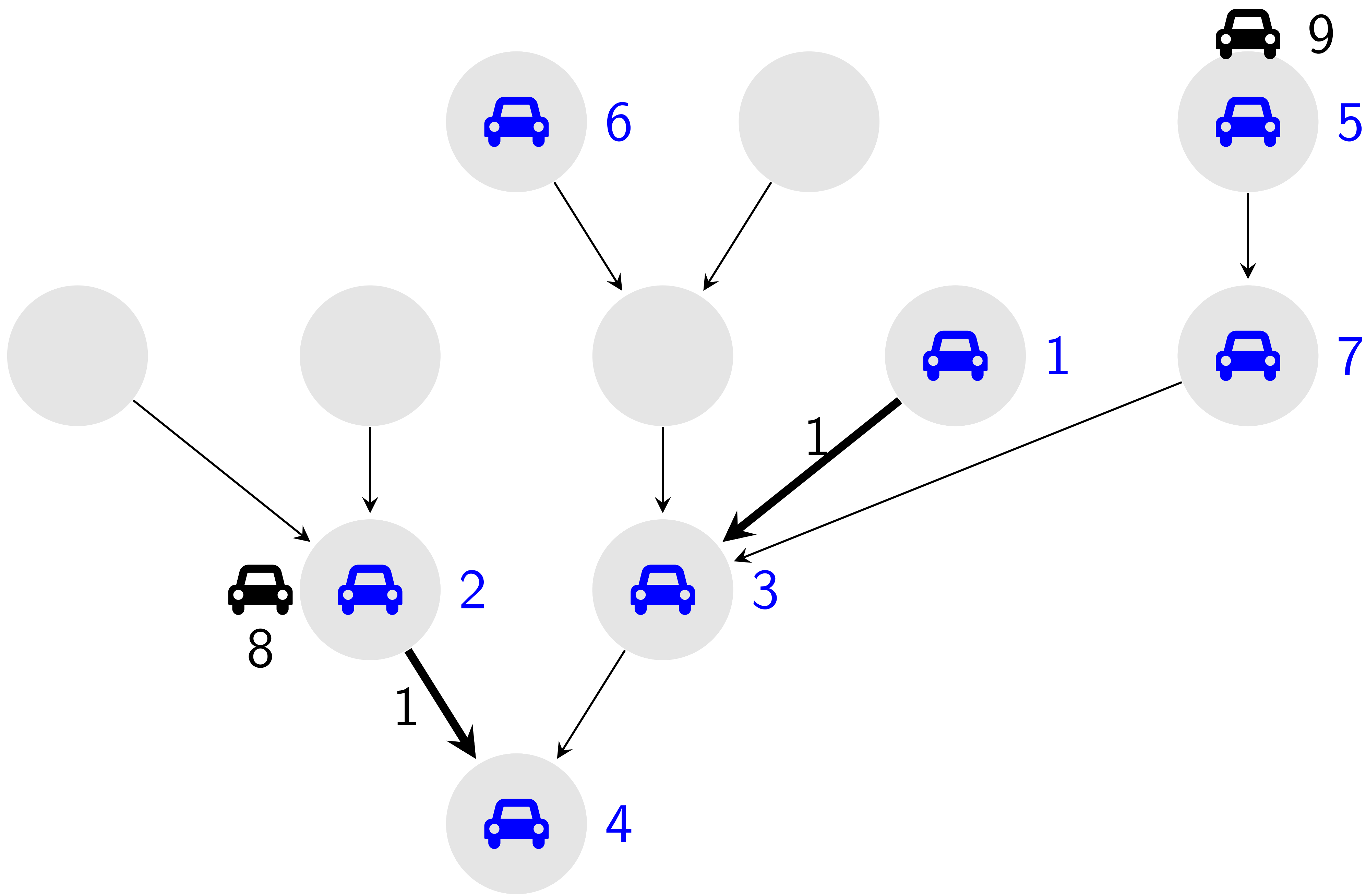

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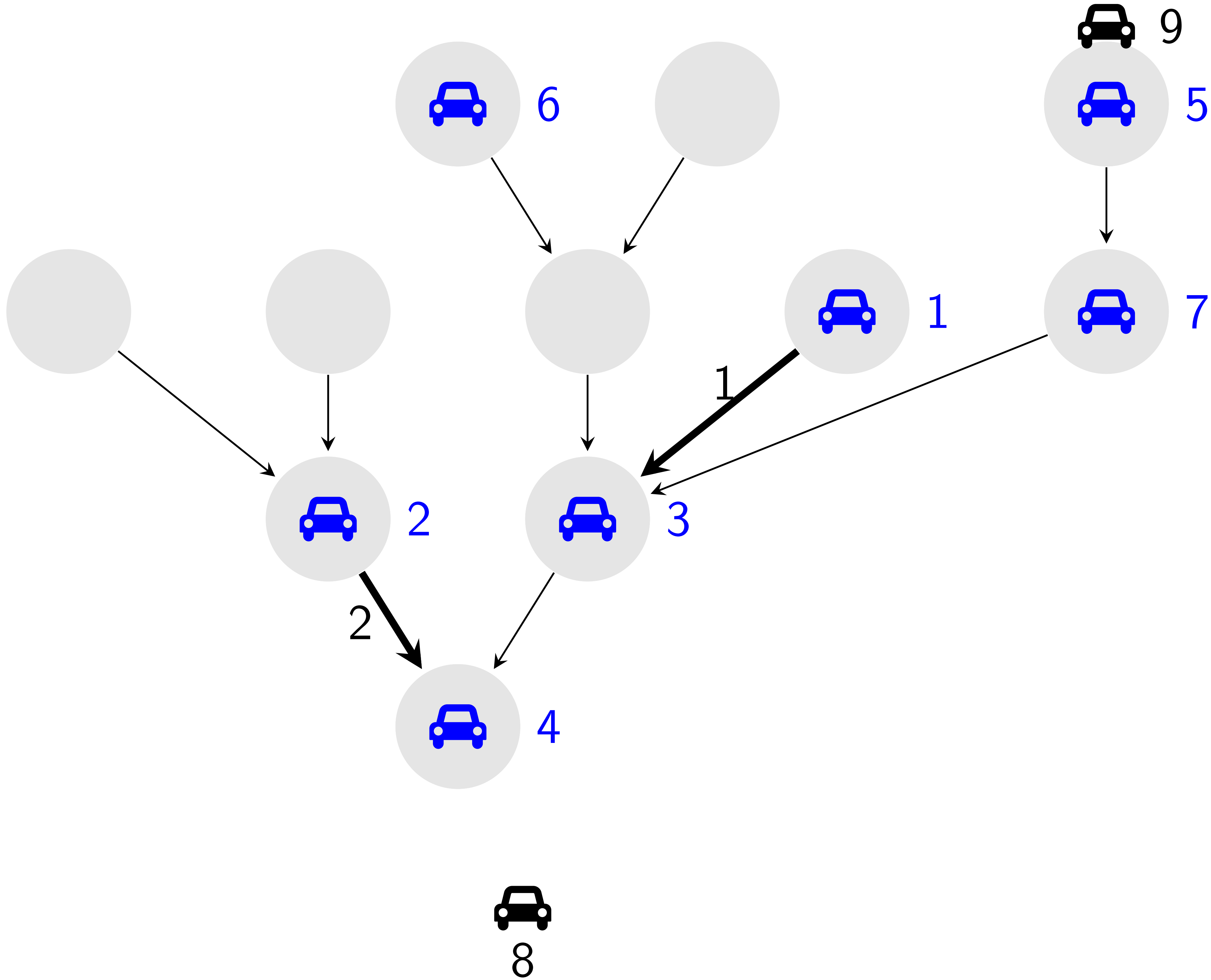

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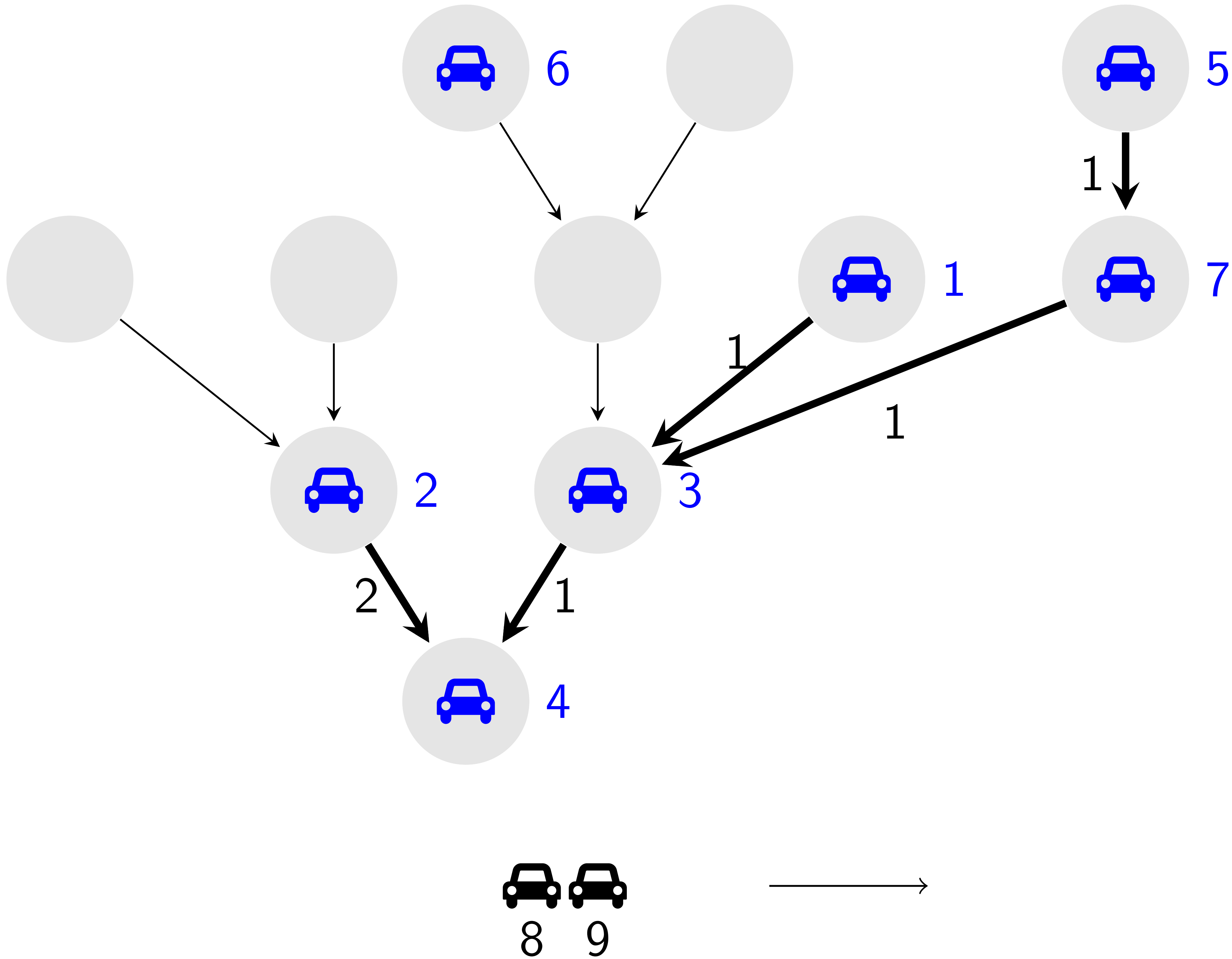

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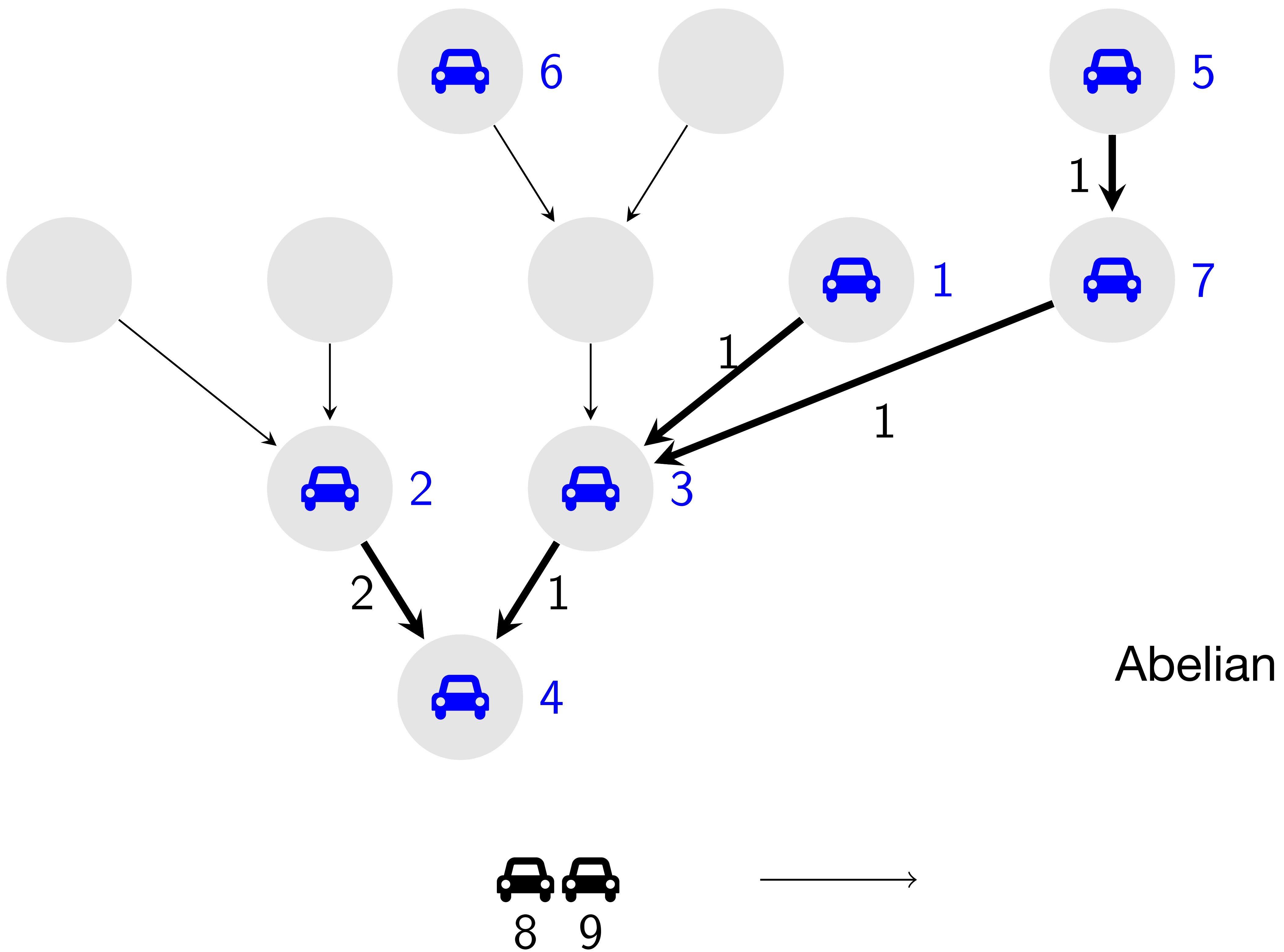
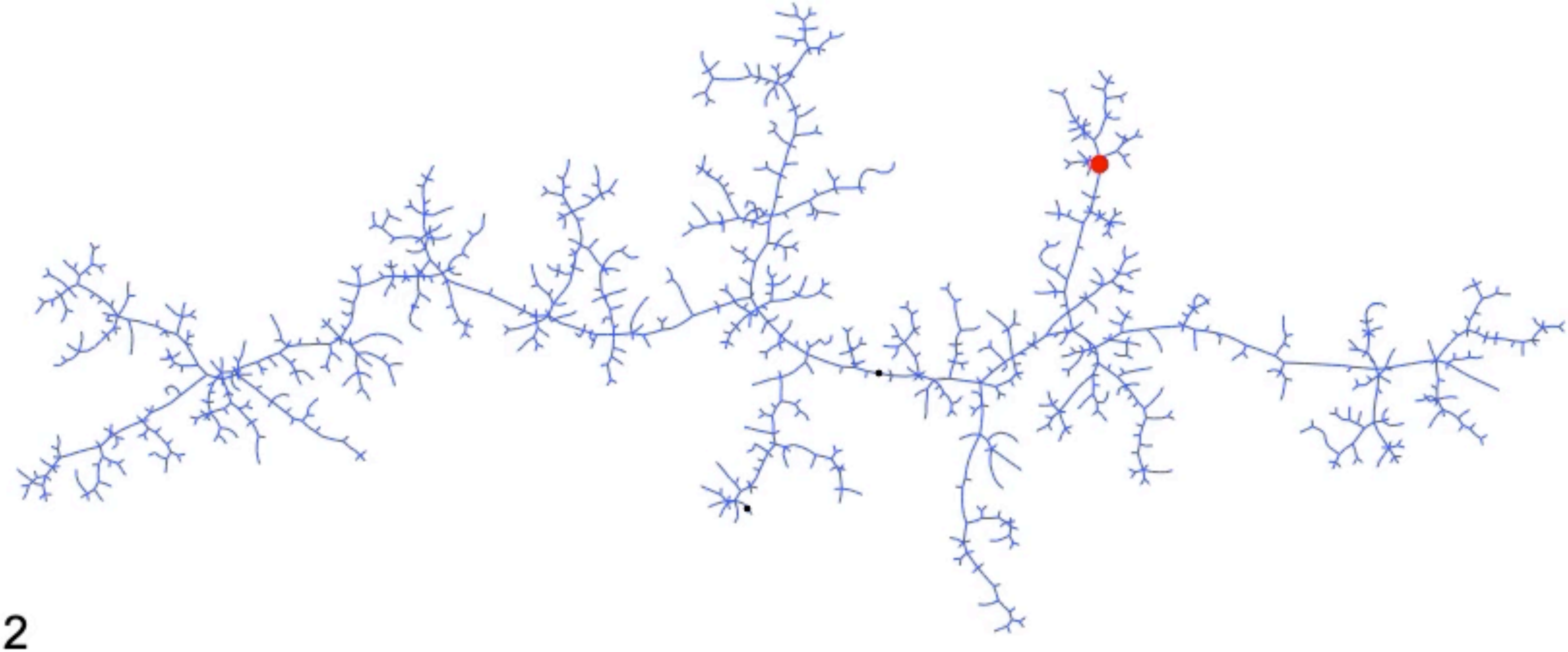


Motivations

Model

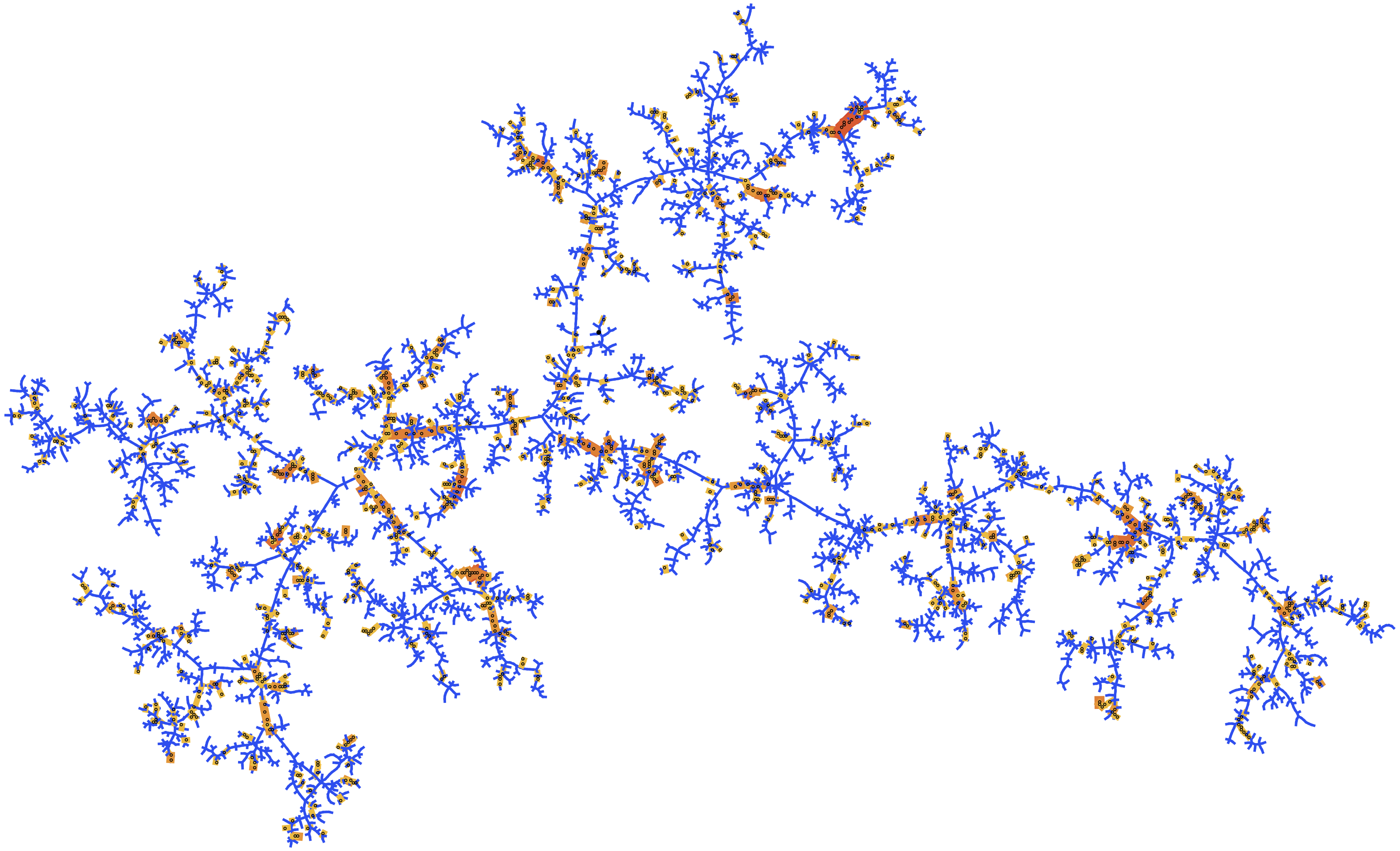
- We choose a (finite or infinite) tree $\mathscr{T}$
- Conditionally on $\mathscr{T}$, the cars arrive one by one uniformly on the vertices.
- **Goal** : study the *flux* of outgoing cars, the size of the *components* of parked cars, their *geometry*…

Movie



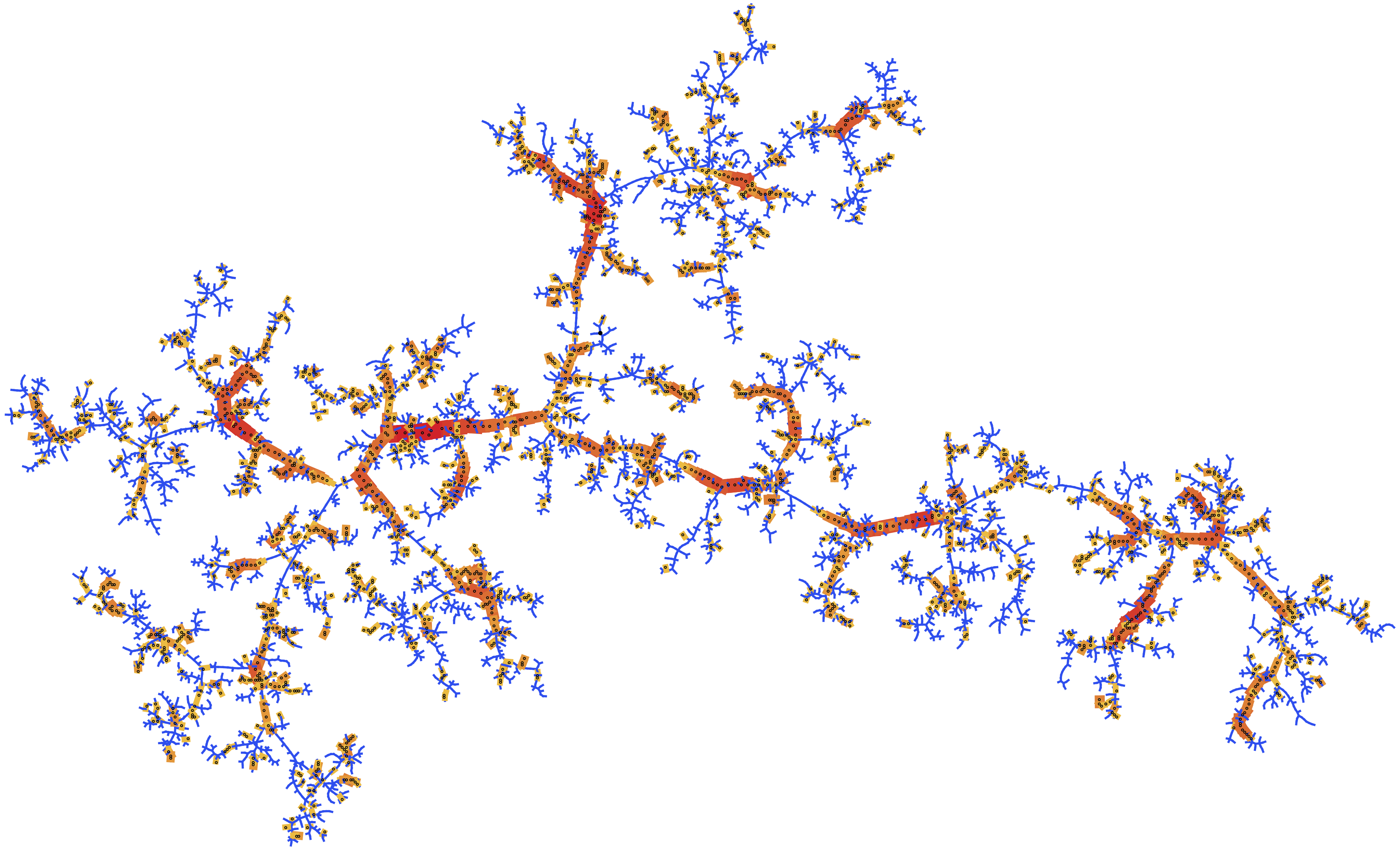
Phase transition

Subcritical regime



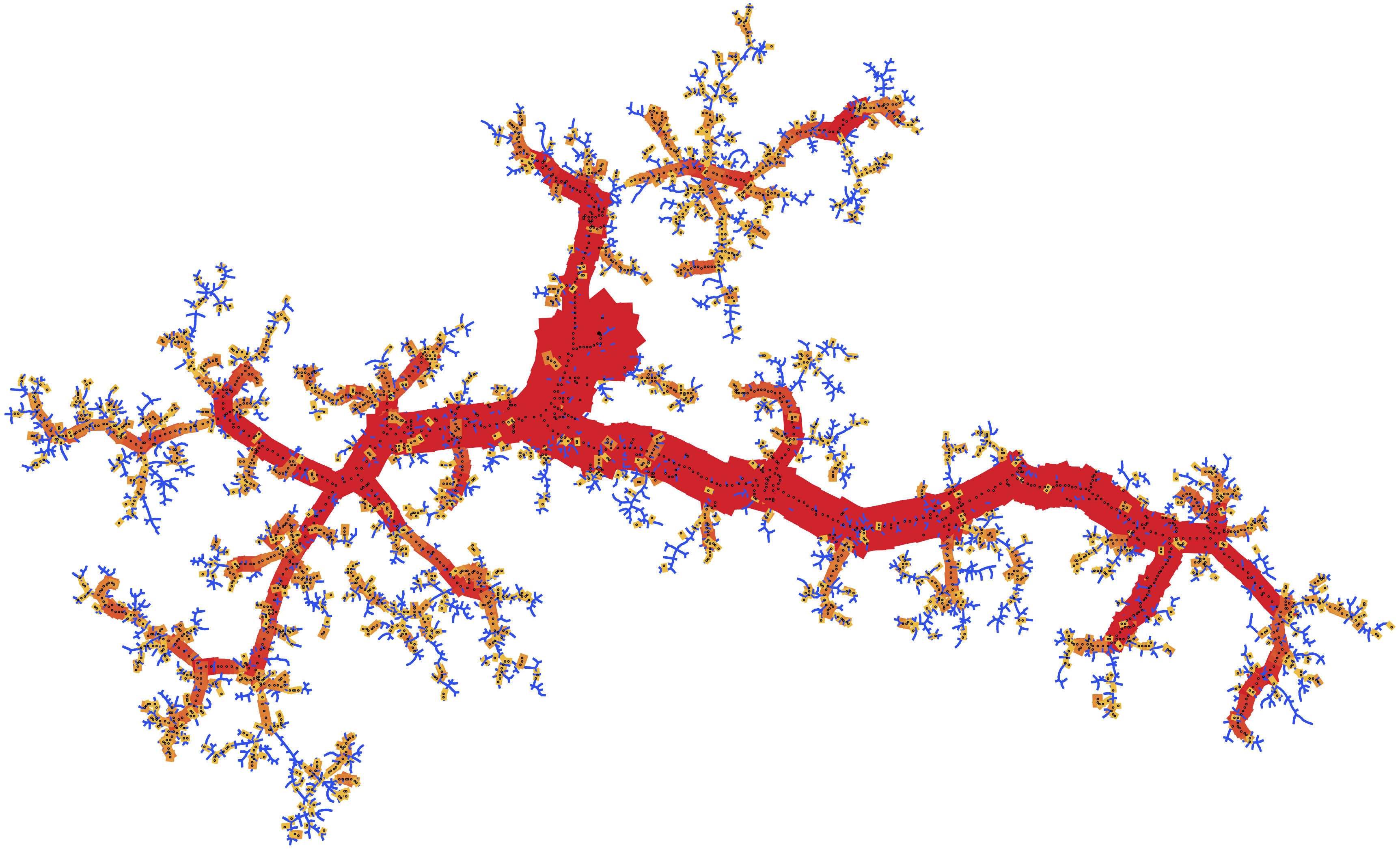
Flux of outgoing cars $= o_{\mathbb{P}}(n)$

Critical regime



Flux of outgoing cars $= o_{\mathbb{P}}(n)$

Supercritical regime



Flux of outgoing cars $= (c + o_{\mathbb{P}}(1))n$

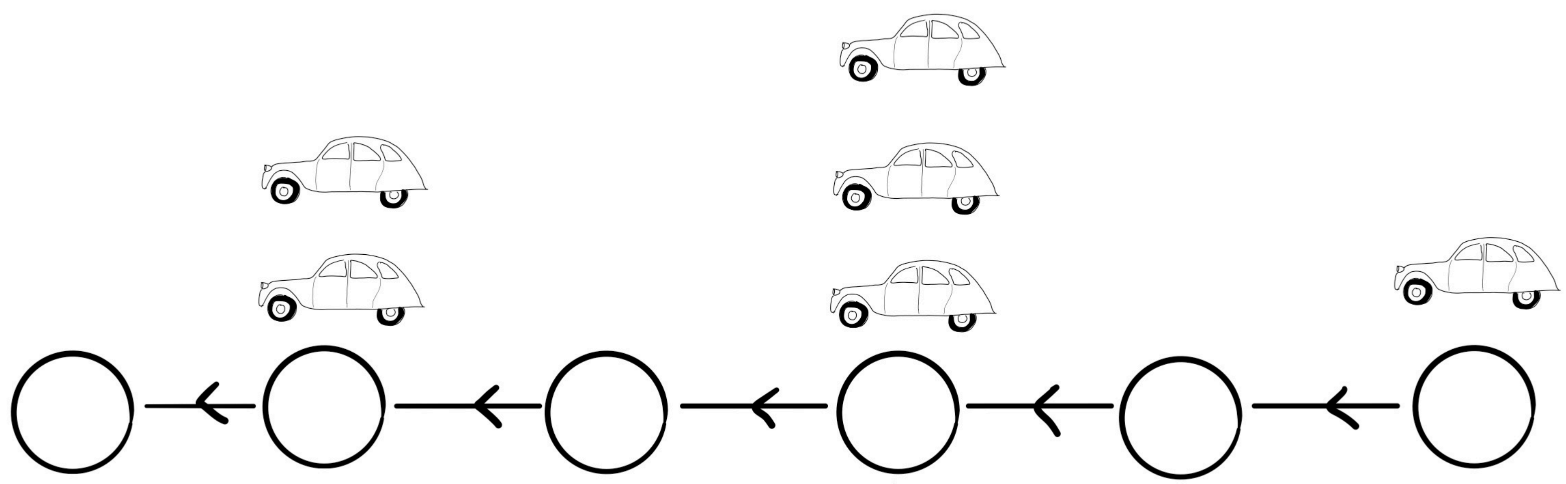
Phase transition

- In the subcritical regime, the flux is sublinear.
- In the supercritical regime, the flux is linear.

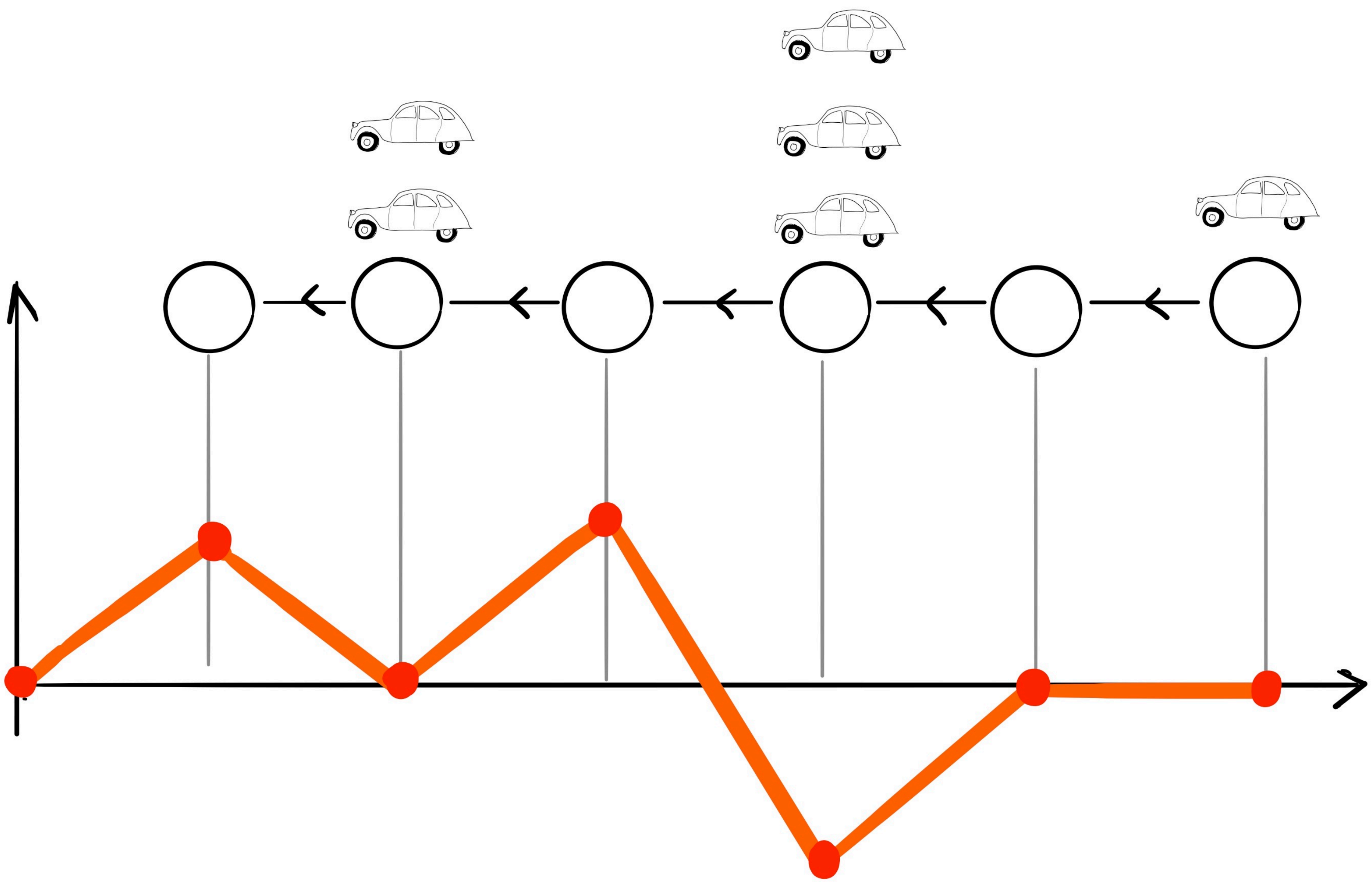
- We take a stochastically increasing family of probability measures $(\mu^\alpha : \alpha \in \mathbb{R})$.
- Critical parameter $\alpha_c \longrightarrow$ phase transition

- Challenge : localize the phase transition and sharp understanding

Case of the line



Case of the line



"Trivial" phase transition

$\longrightarrow \alpha = 1$ whatever the car arrival law

History

- **1960 :** Konheim & Weiss : Line
- **2015 :** Lackner & Panholzer : Uniform Cayley trees
- **2016 :** Jones : Uniform binary trees
- **2016 :** Goldschmidt and Przykucki : Uniform Cayley trees (bis)
- **2019 :** Chen and Goldschmidt : Uniform plane trees
- **2019 :** Curien & Hénard (voir aussi C.): Conditionned Bienaymé—Galton—Watson trees
- **2021 :** C. & Curien: Uniform Cayley trees (ter)
- **2022 :** Aldous, C., Curien & Hénard: Infinite binary tree

…

Three techniques for parking on a tree

Local limit and
differential equations

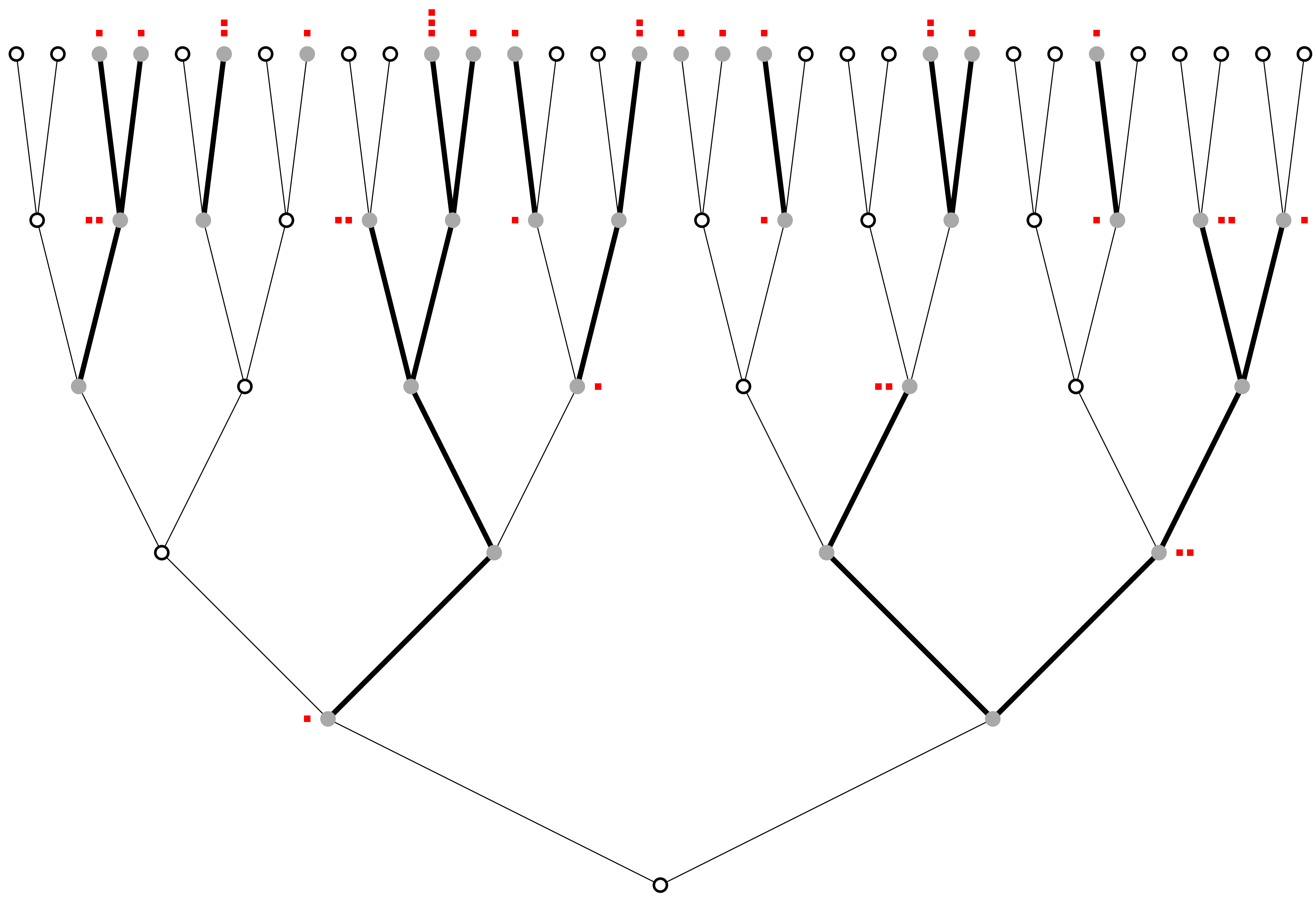
Coupling with the
Erdős—Rényi random graph

Combinatorial decomposition
and "à la Tutte" recursion

Infinite binary tree:

Combinatorial decomposition and "à la Tutte" recursion

Infinite binary tree



- I.i.d. car arrivals with law μ. We denote by G its generating function

$$G(t) = \sum_{k \geq 0} \mu_k t^k$$

Localisation of the phase transition

Theorem (Aldous, C., Curien & Hénard, 2022) :

Suppose that there exists

$$t_c = \min\{t \geq 0,\ 2(G(t) - tG'(t))^2 = t^2 G(t)G''(t)\}\,.$$

The parking process is subcritical if and only if

$$(t_c - 2)G(t_c) \geq t_c(t_c - 1)G'(t_c)\,.$$

In general, the parameter t_c exists.

Examples

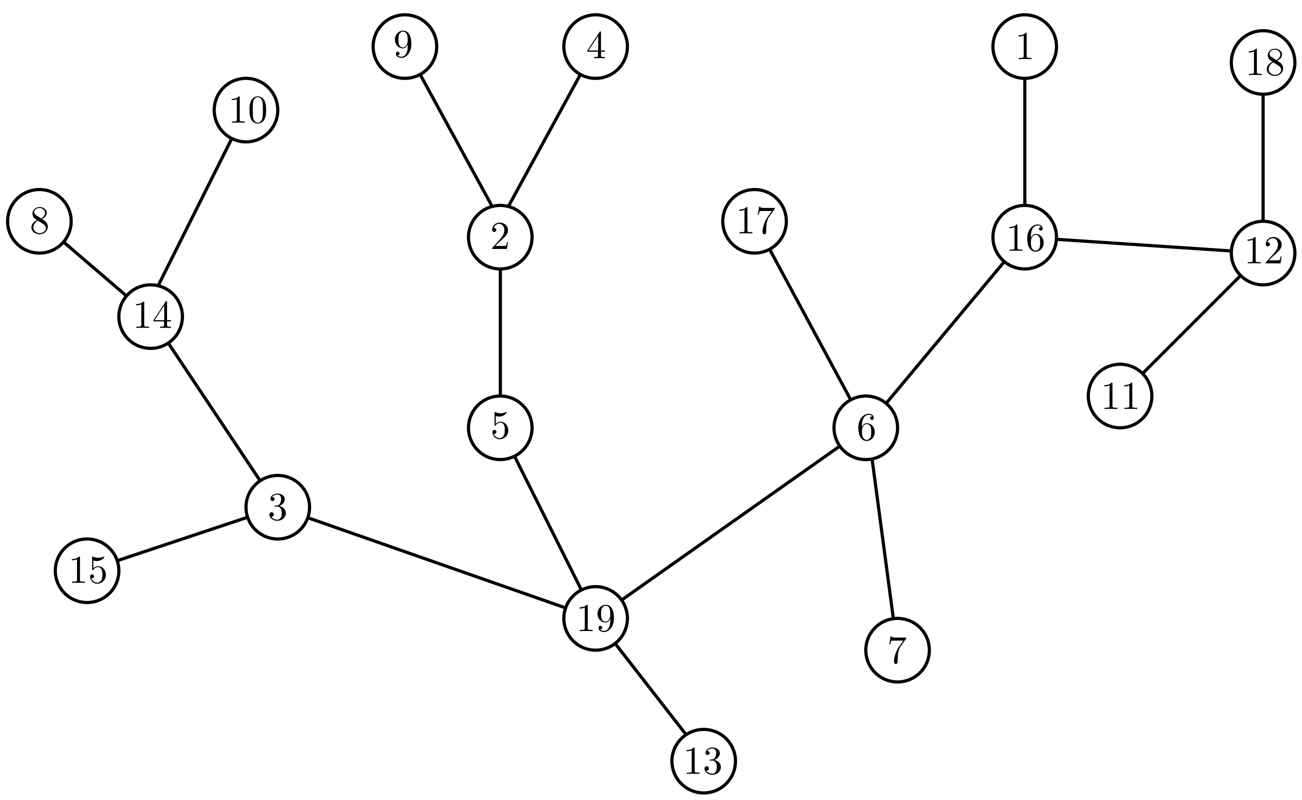
Car arrivals	Critical value α_c
Binary 0/2 $\mu^{\alpha} = (1 - \frac{\alpha}{2})\delta_0 + \frac{\alpha}{2}\delta_2$	$\frac{1}{14}$
Poisson $G_{\alpha}(t) = \exp\left(t(a-1)\right)$	$3 - 2\sqrt{2}$
Geometric $G_{\alpha}(t) = \frac{1}{1+\alpha-\alpha t}$	$\frac{1}{8}$

Cayley trees :

Coupling
with the Erdős—Rényi random graph

Model

- We take a uniform Cayley tree $\mathscr{T}_n$ with n vertices.



- $m = \alpha n$ cars arrive sequentially, uniformly and independently on the vertices of $\mathscr{T}_n$.

Theorem (C. & Curien, 2021):

We can couple the parking process and the Erdős—Rényi random graph model.

The phase transition occurs at $\alpha_c = 1/2$.

In the critical window, that is when

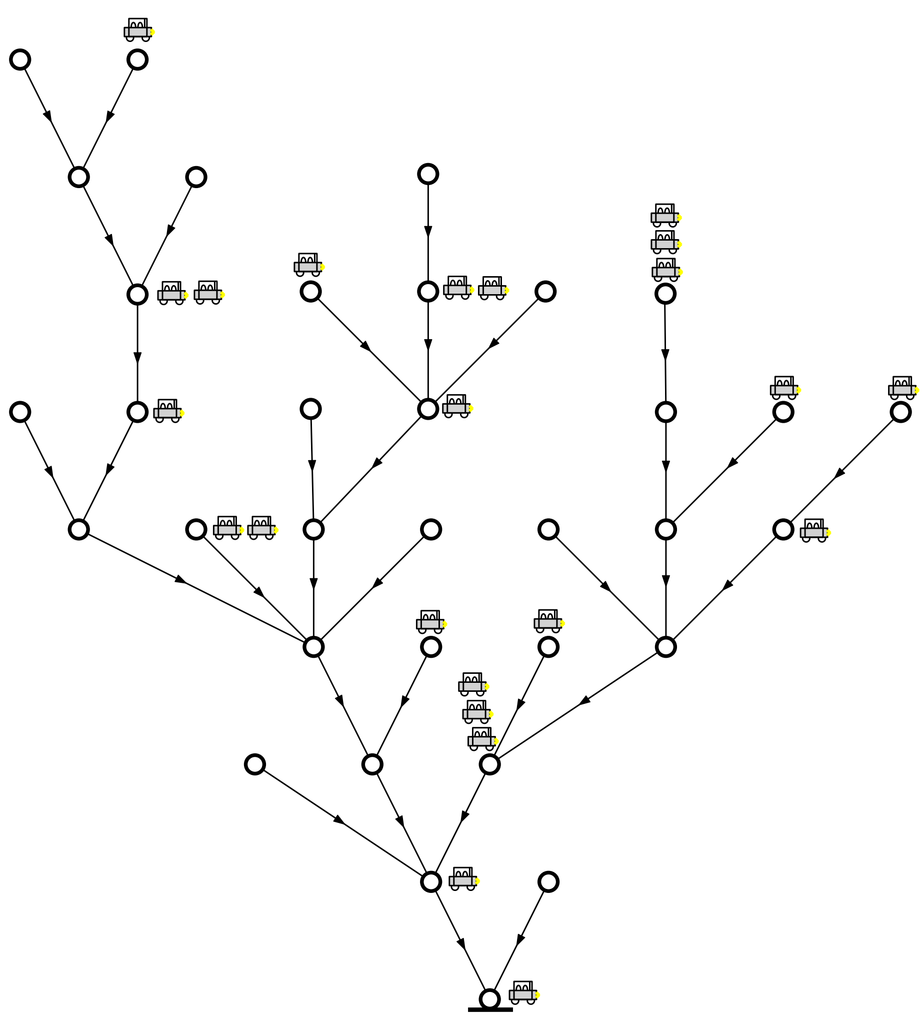
$$m = \left\lfloor \frac{n}{2} + \frac{\lambda}{2} \cdot n^{2/3} \right\rfloor, \text{ with } \lambda \in \mathbb{R},$$

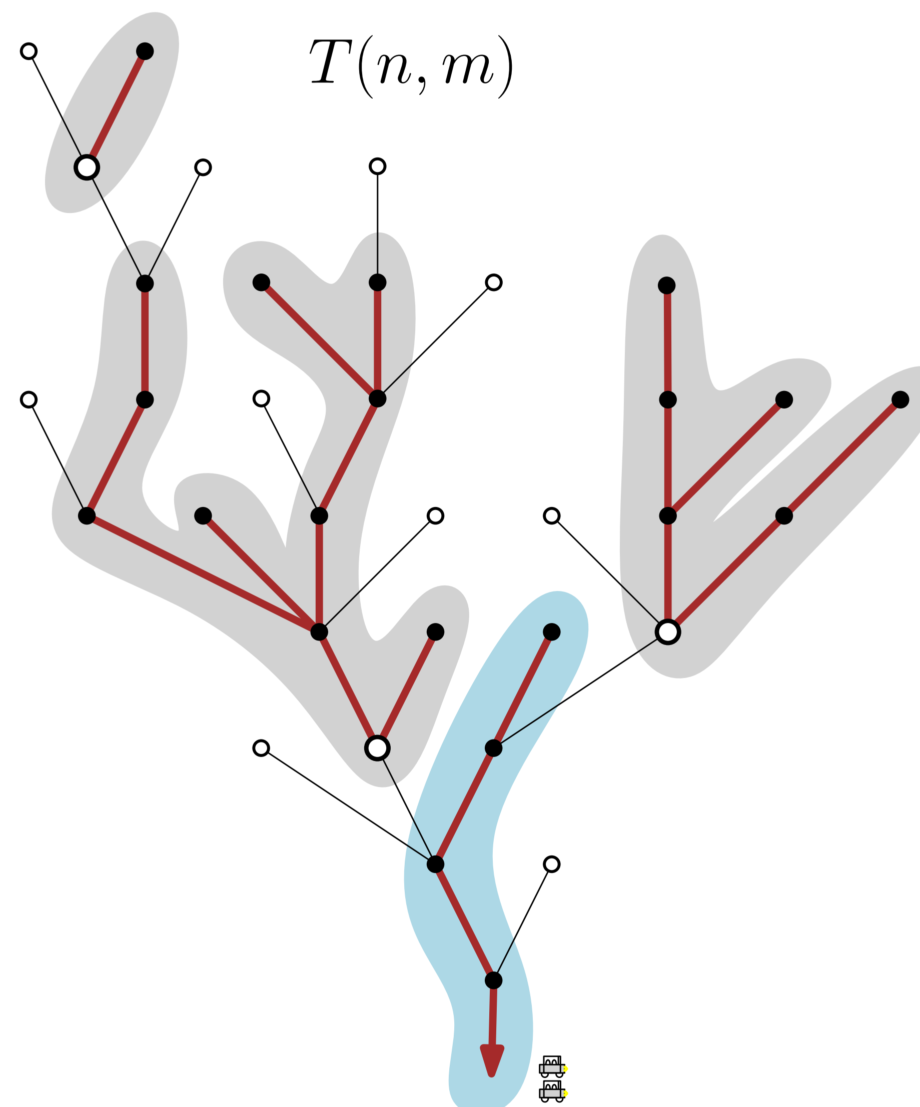
the clusters of parked cars have size of order $n^{2/3}$, and the flux of outgoing cars is of order $n^{1/3}$.

Markovian exploration

Idea : Consider the underlying tree as unknown and discover its edges step-by-step while parking the cars.

$T(n, m)$ forest containing the edges emanating from the occupied spots.

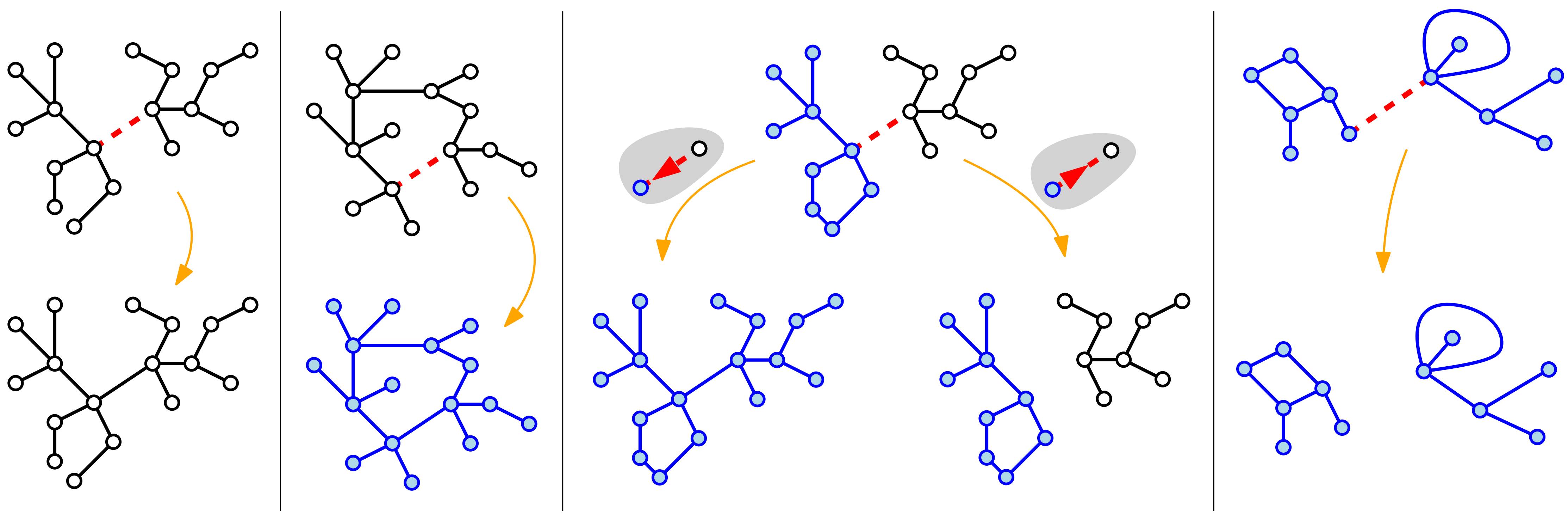




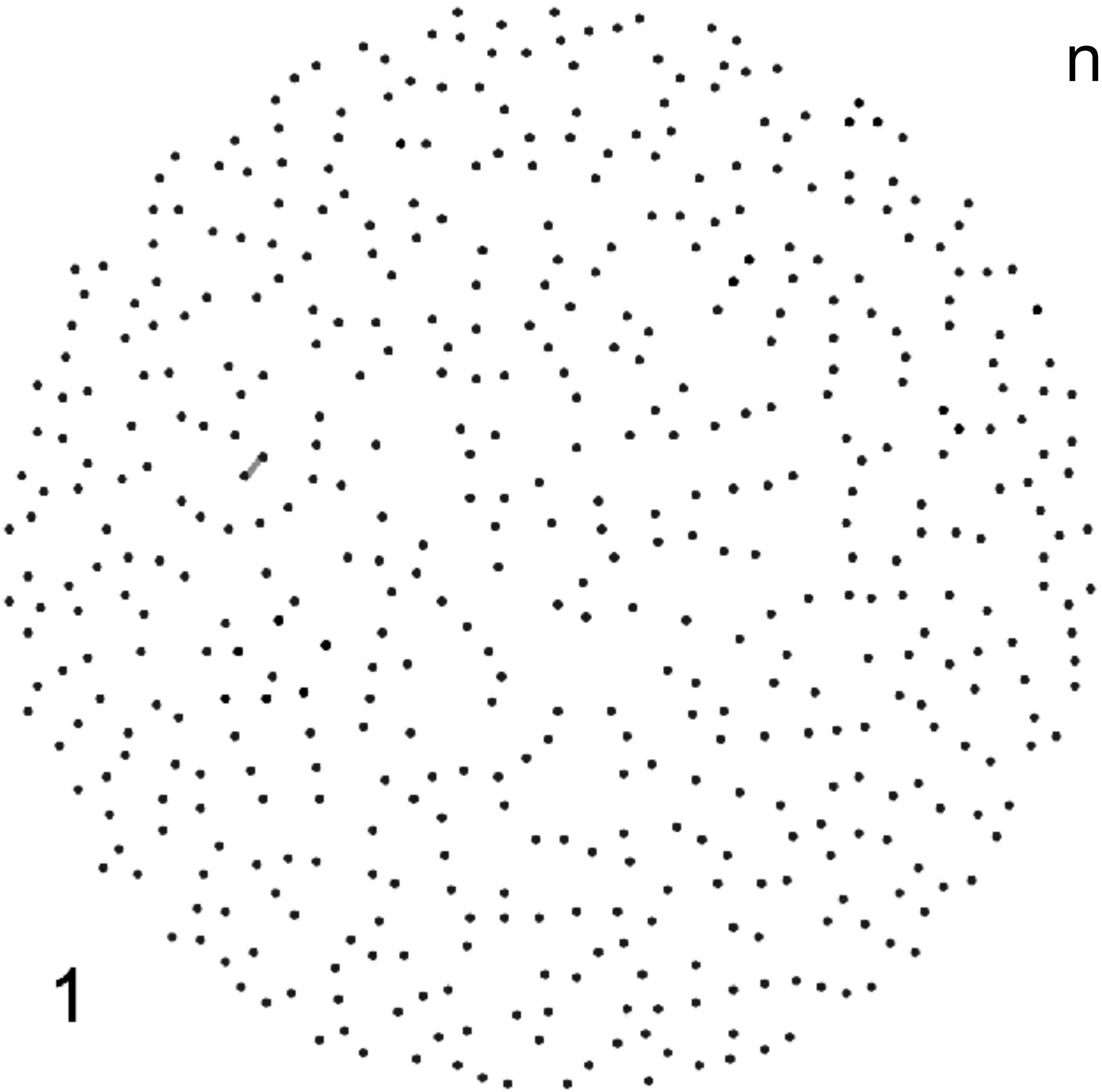
$T(n, m)$ is a Markov chain.

The *frozen* Erdős—Rényi random graph

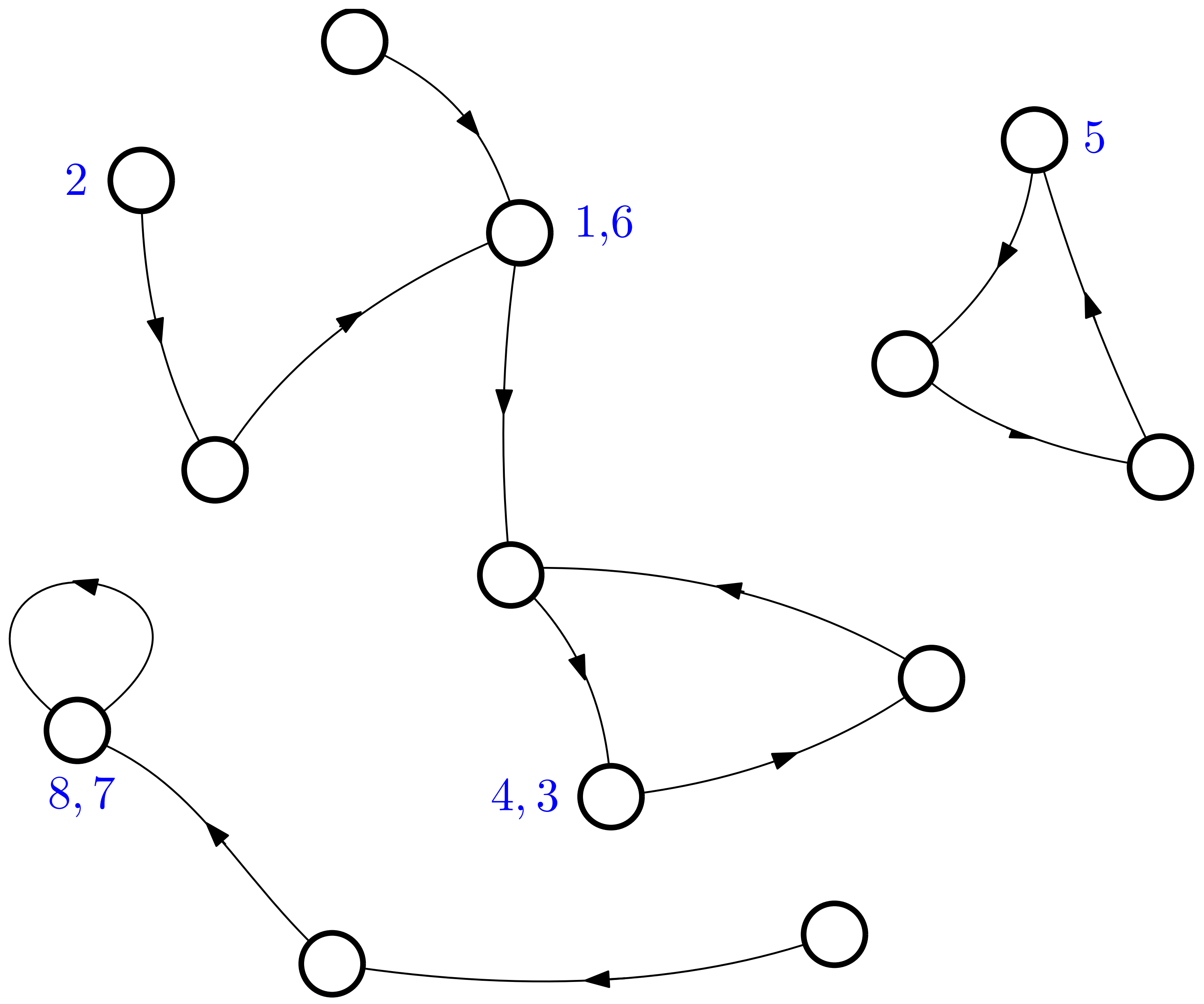
- Labeled vertices $\{1,2,\ldots,n\}$.
- For $i \geq 1$, oriented edge $\overrightarrow{E}_i = (X_i, Y_i)$ where X_i and Y_i are two uniform vertices in $\{1,2,\ldots,n\}$ and E_i its unoriented version.
- The ***standard*** Erdős—Rényi process $G(n, m)$ with edges $\{\{E_i : 1 \leq i \leq m\}\}$
- The ***frozen*** Erdős—Rényi process $F(n, m)$, white or blue vertices. We start with the forest of n white isolated vertices $F(n,0)$. Then,



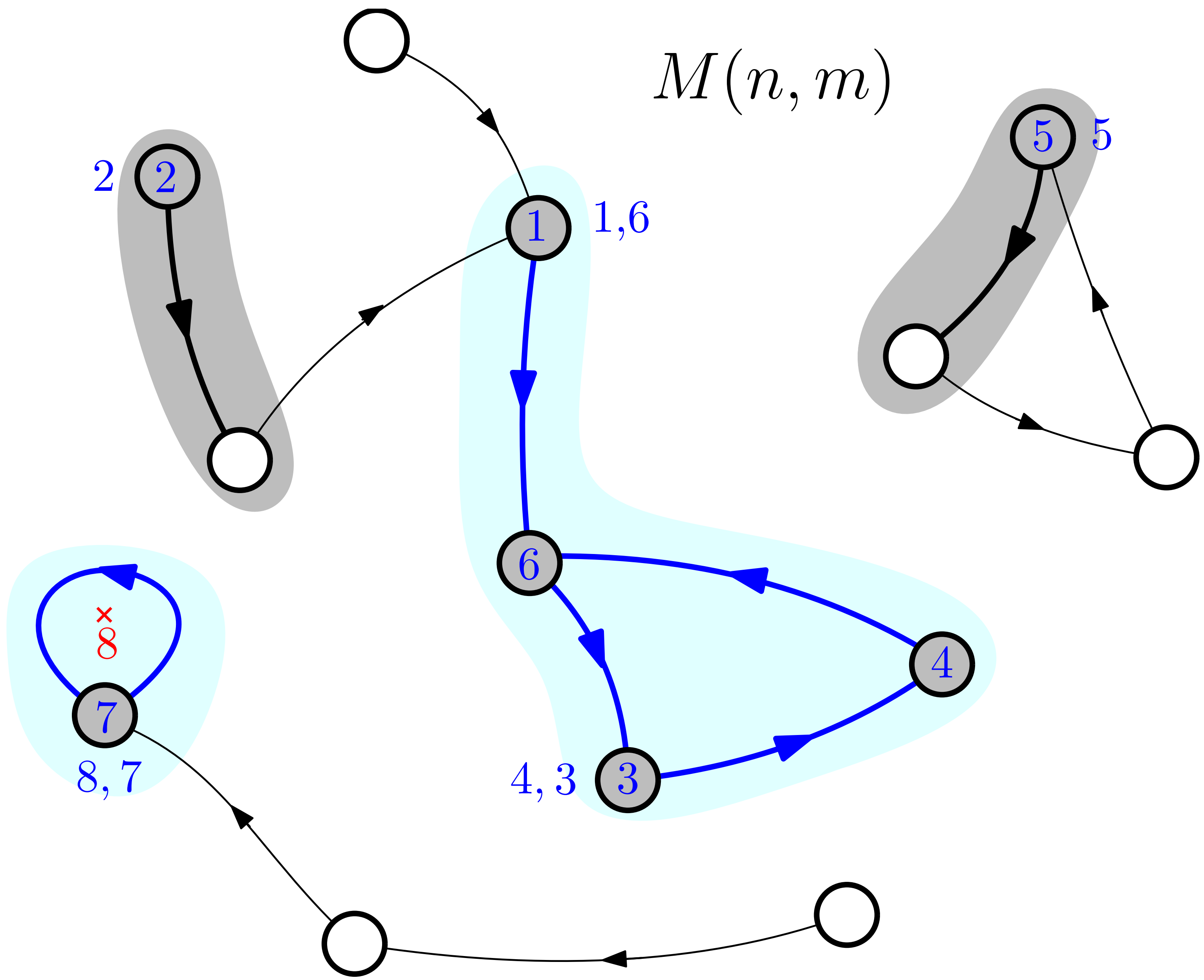
Movie : the frozen Erdős—Rényi random graph



Parking on a mapping



Parking on a mapping

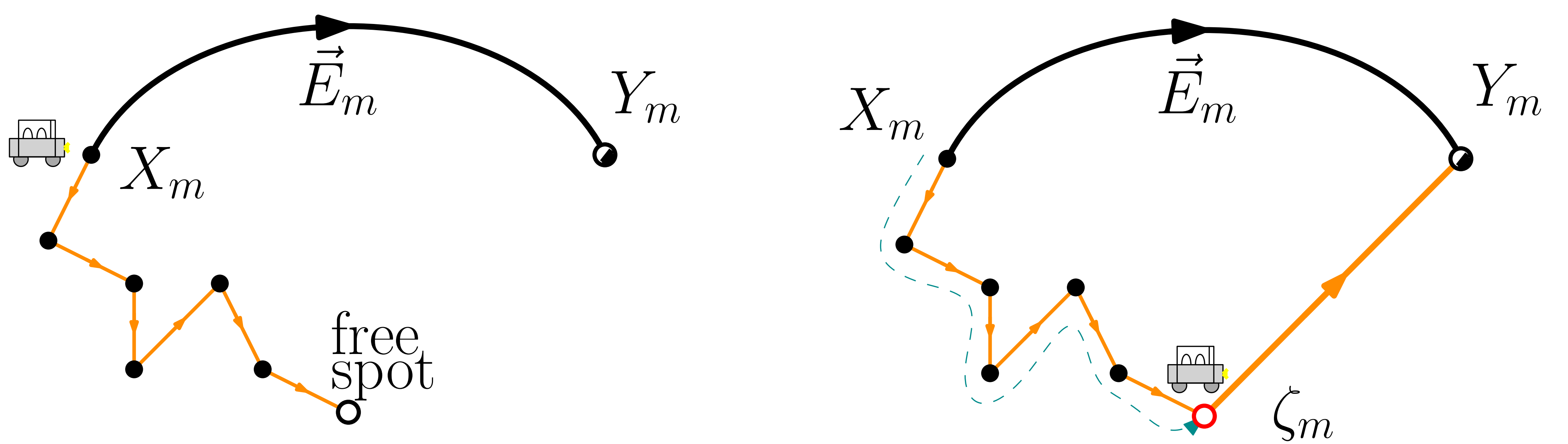


Coupling parking and frozen ER

- We simultaneously built

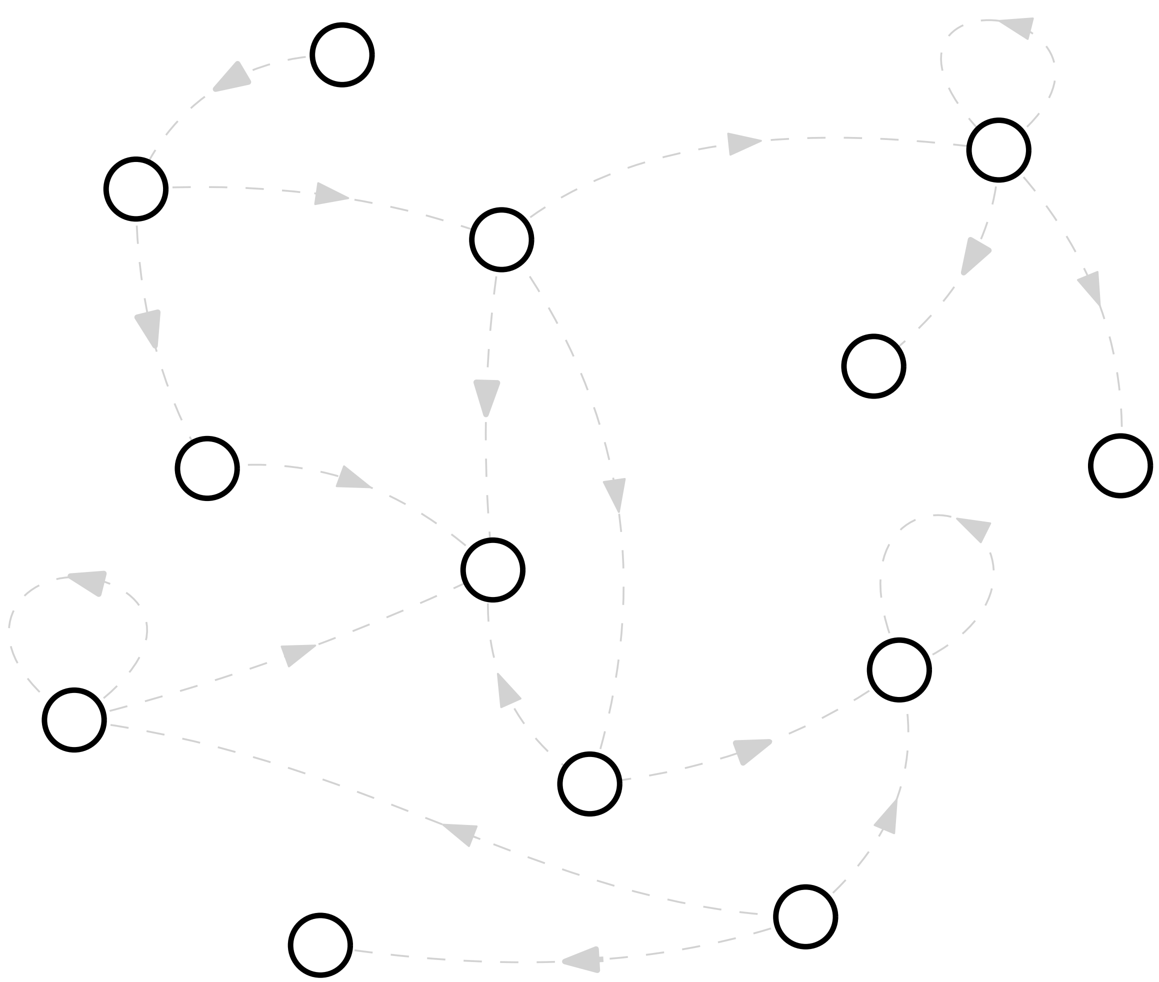
$$(F(n,m) : m \geq 0) \quad \text{and} \quad (M(n,m) : m \geq 0)$$

- Rule :

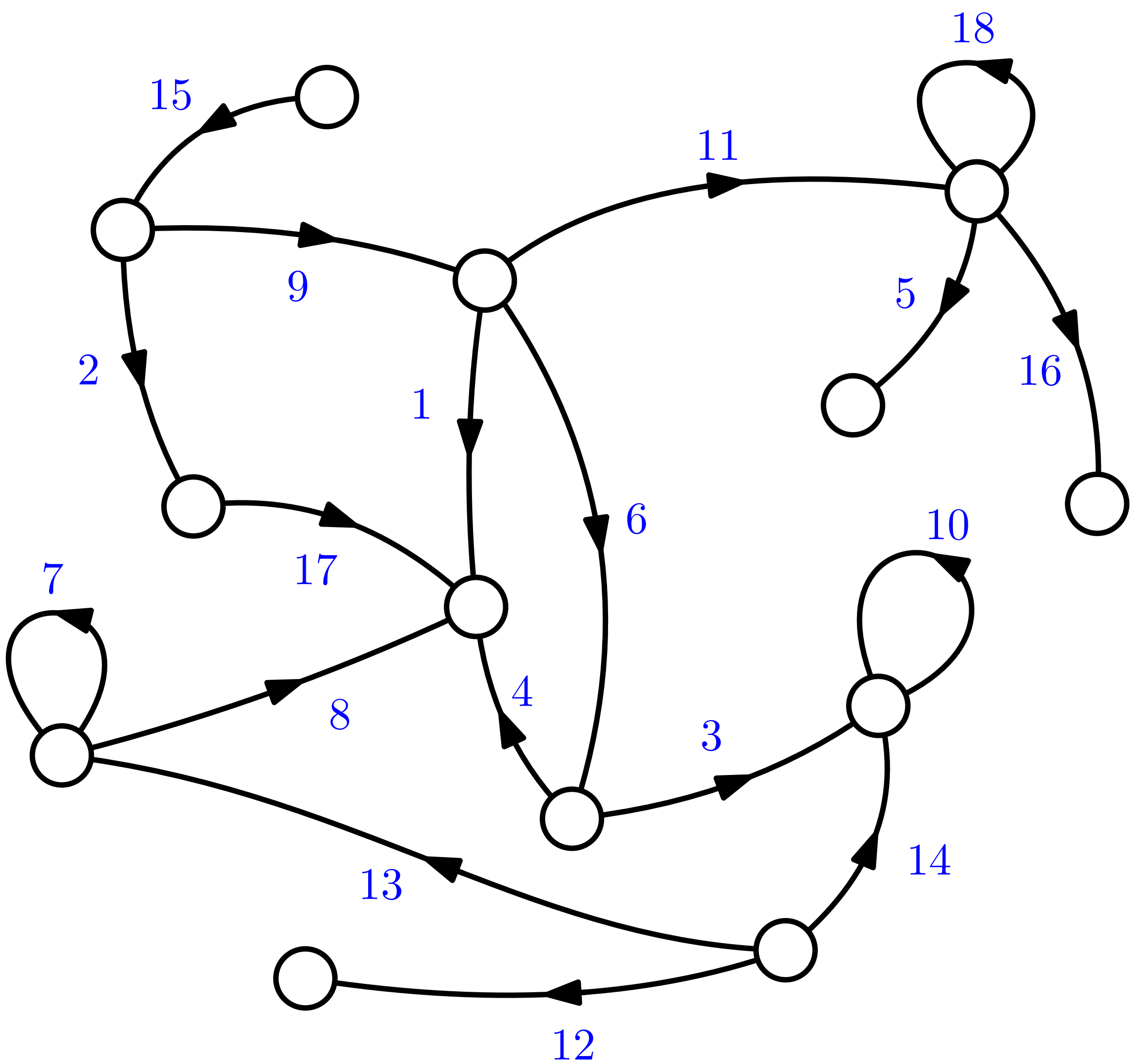


Coupling

Parking on a mapping



Erdős—Rényi



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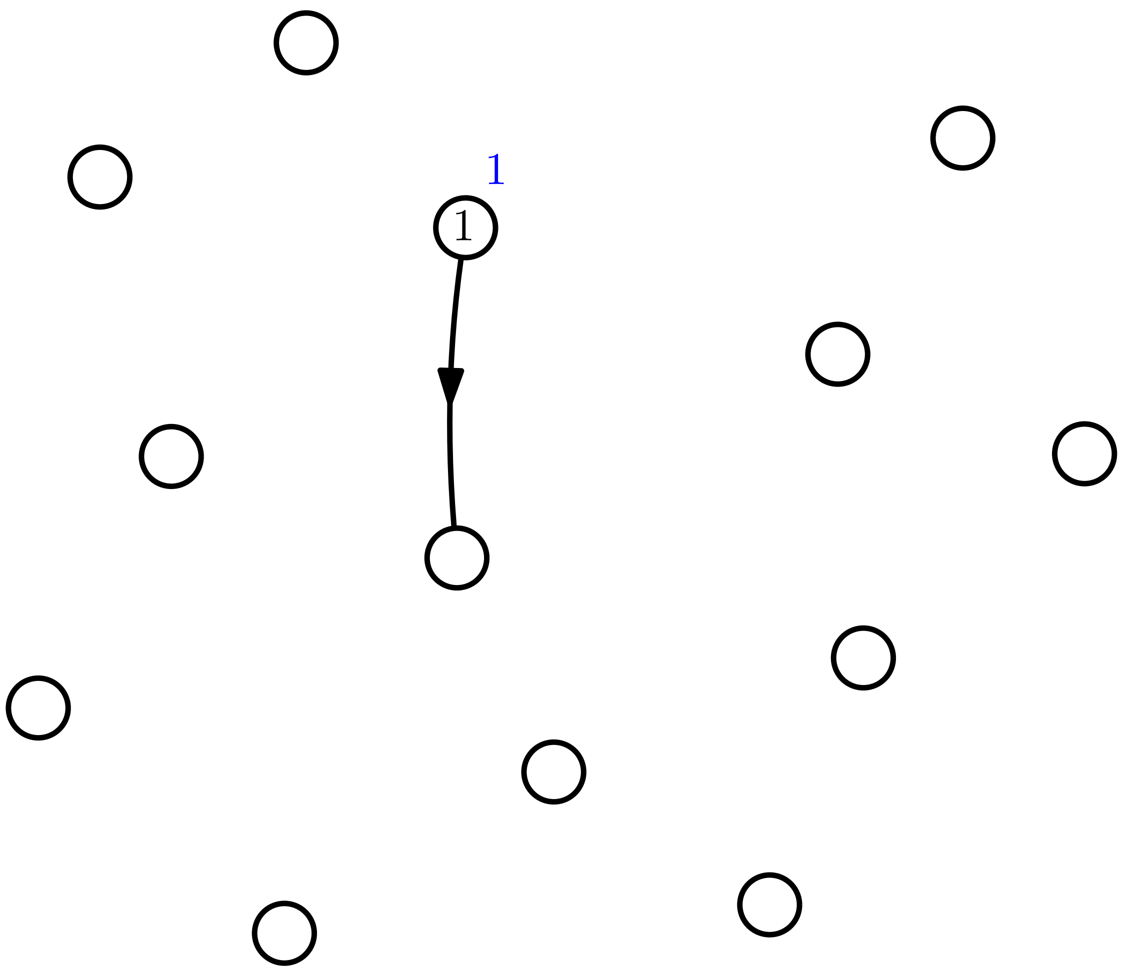
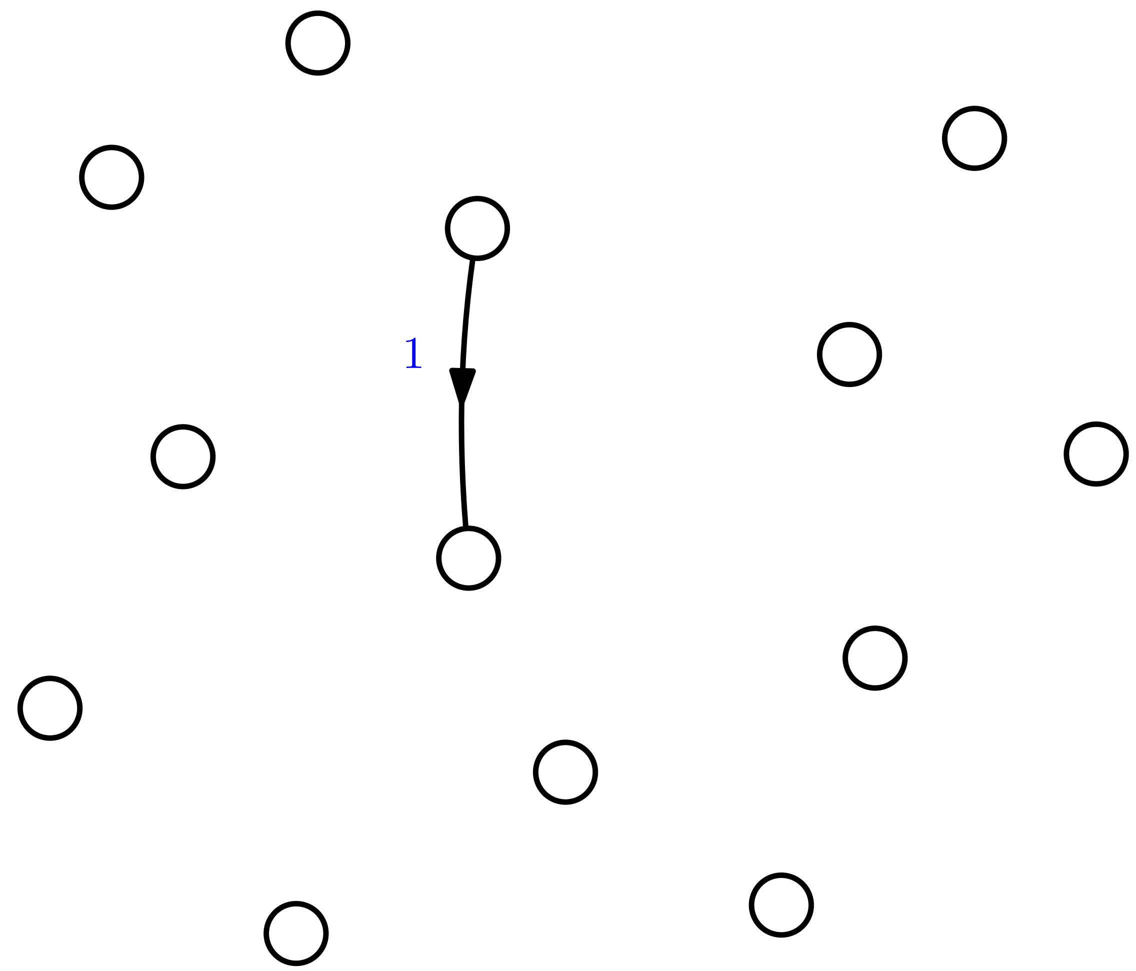


Frozen Erdős—Rényi



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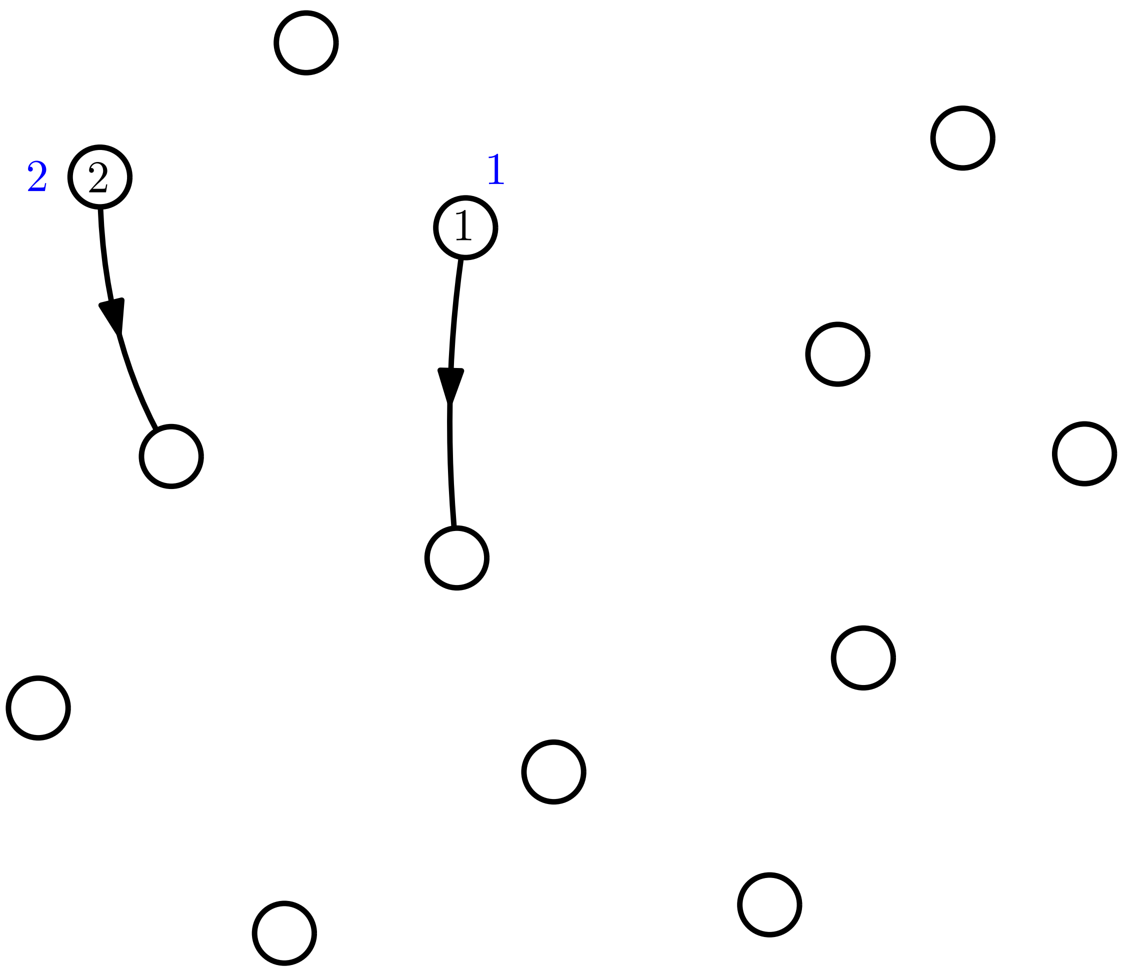
Parking on a mapping



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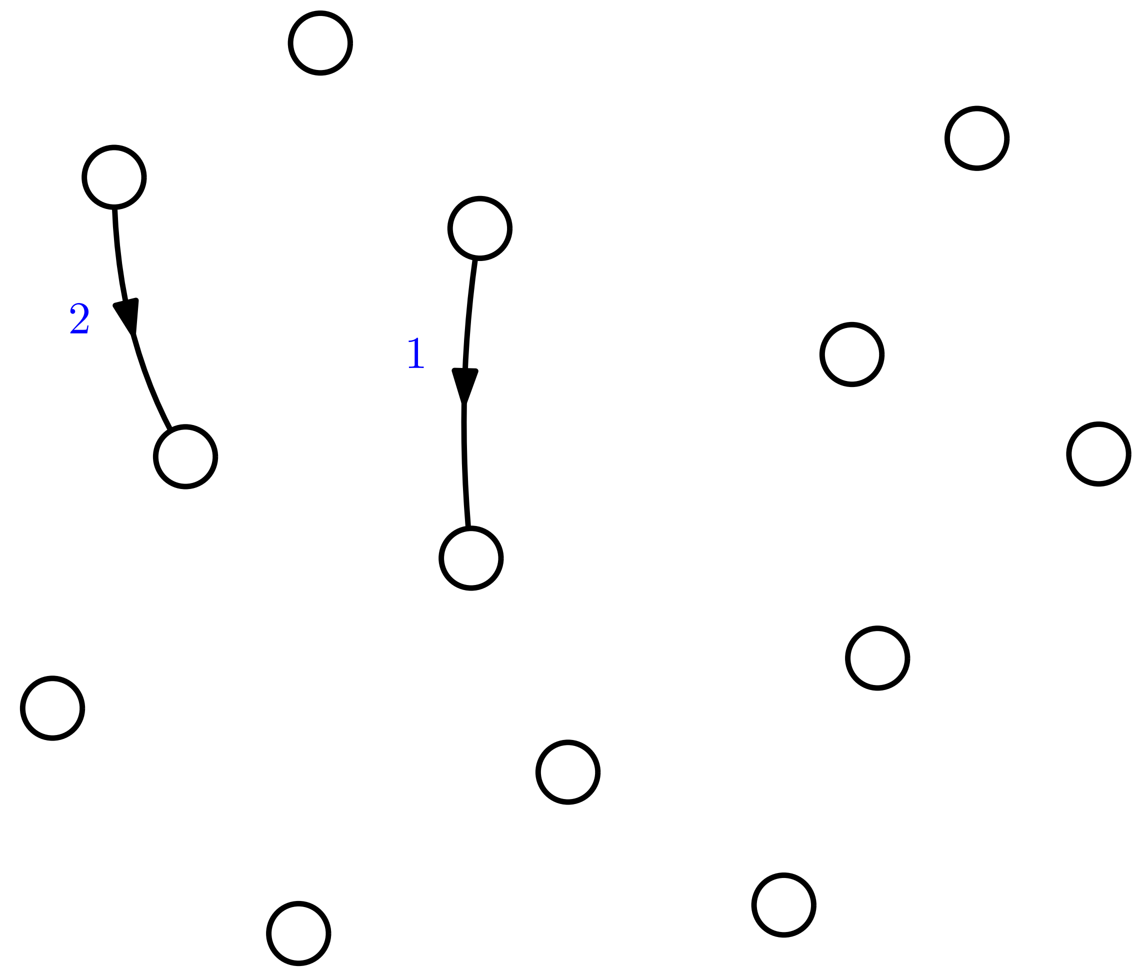

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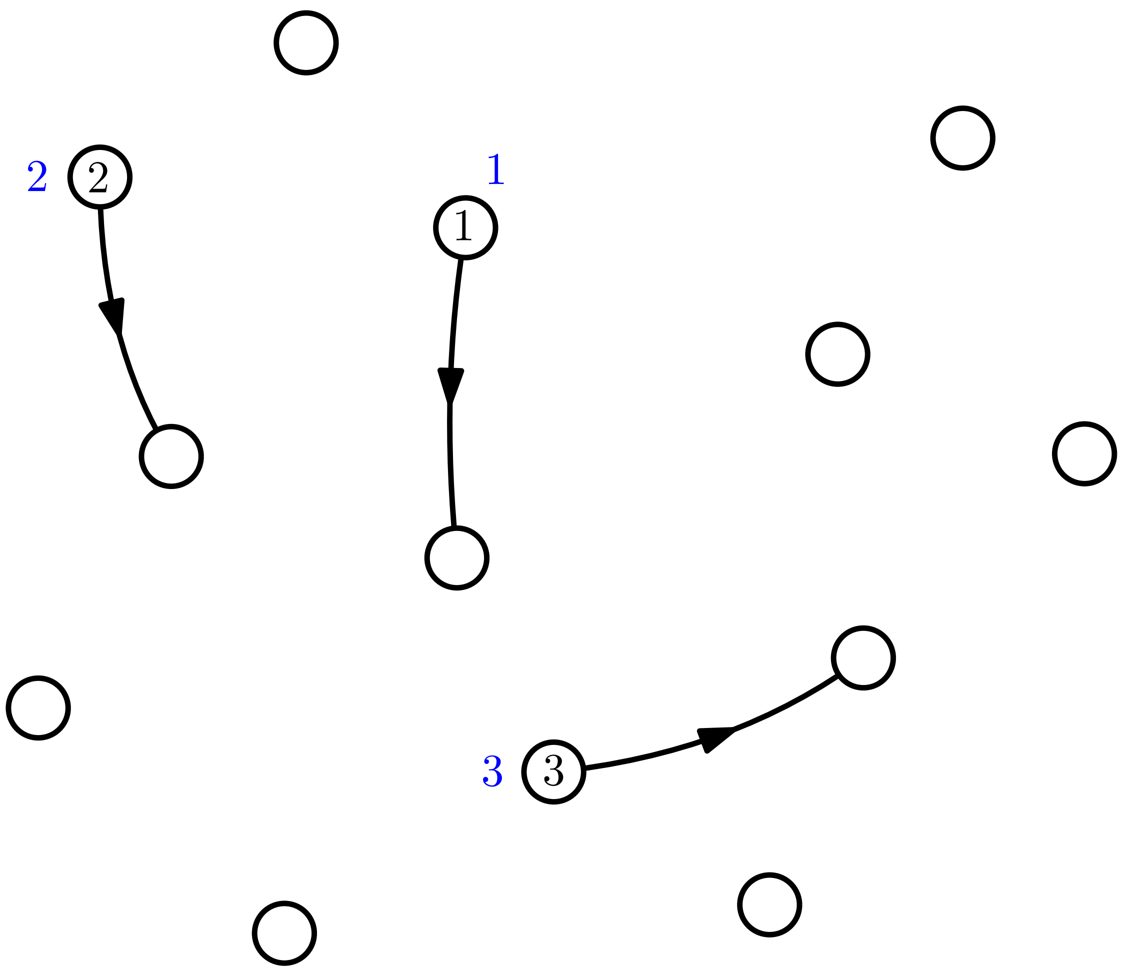
Parking on a mapping



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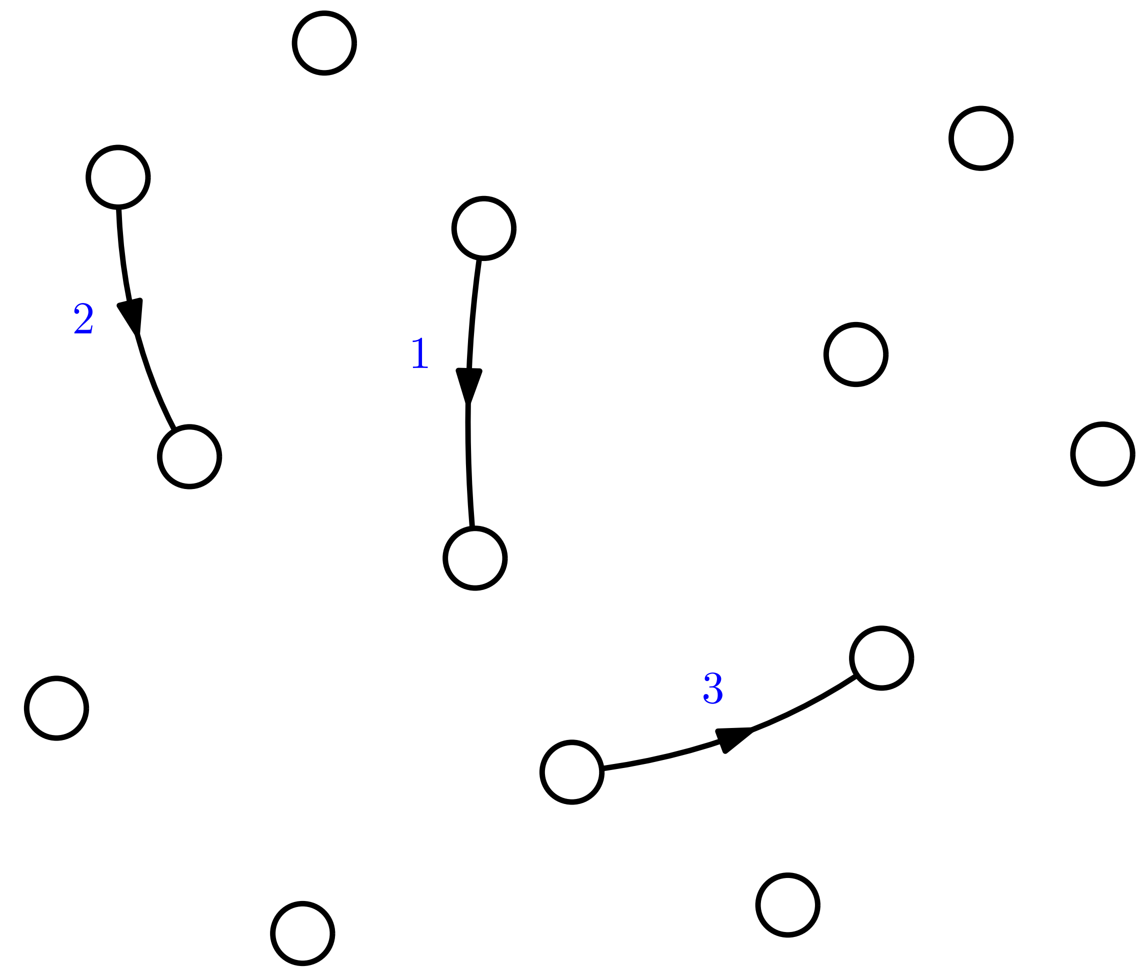

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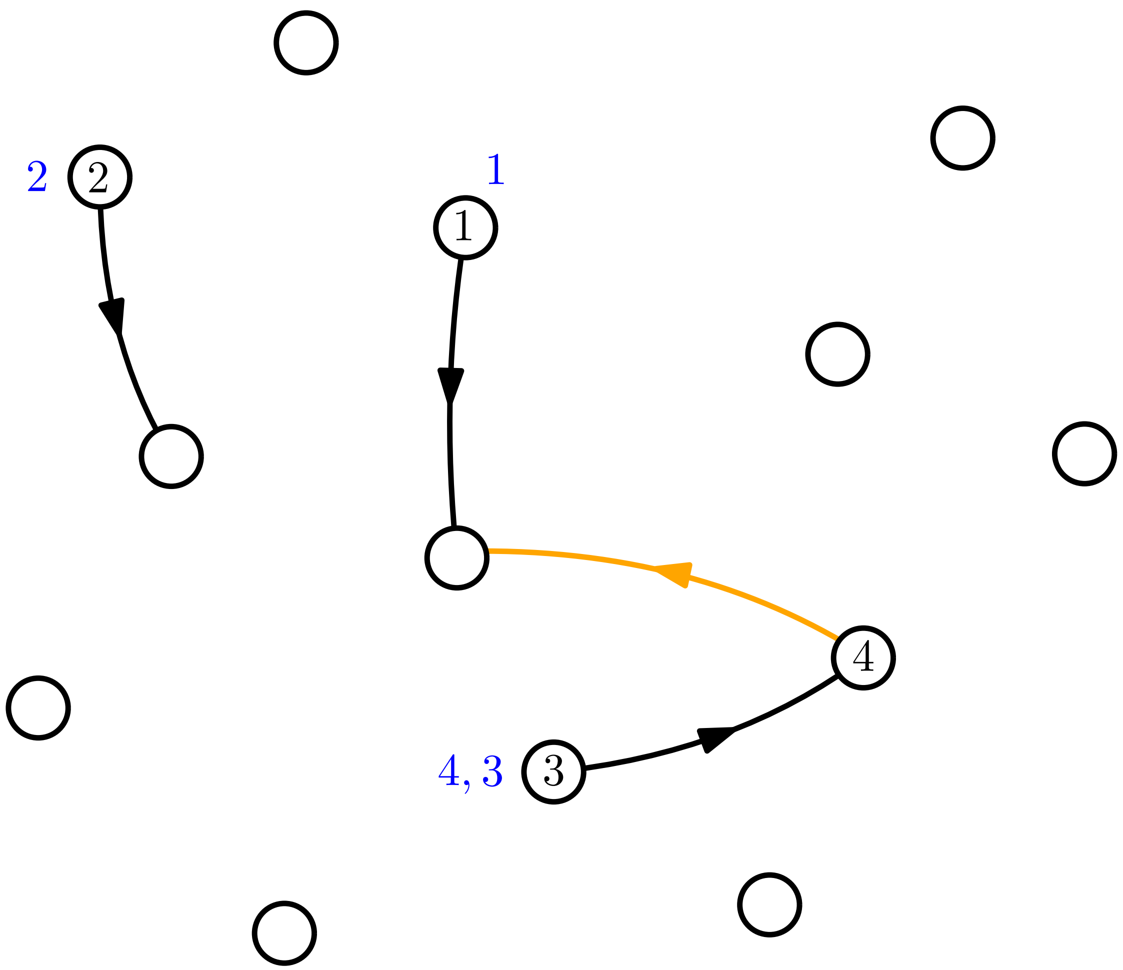

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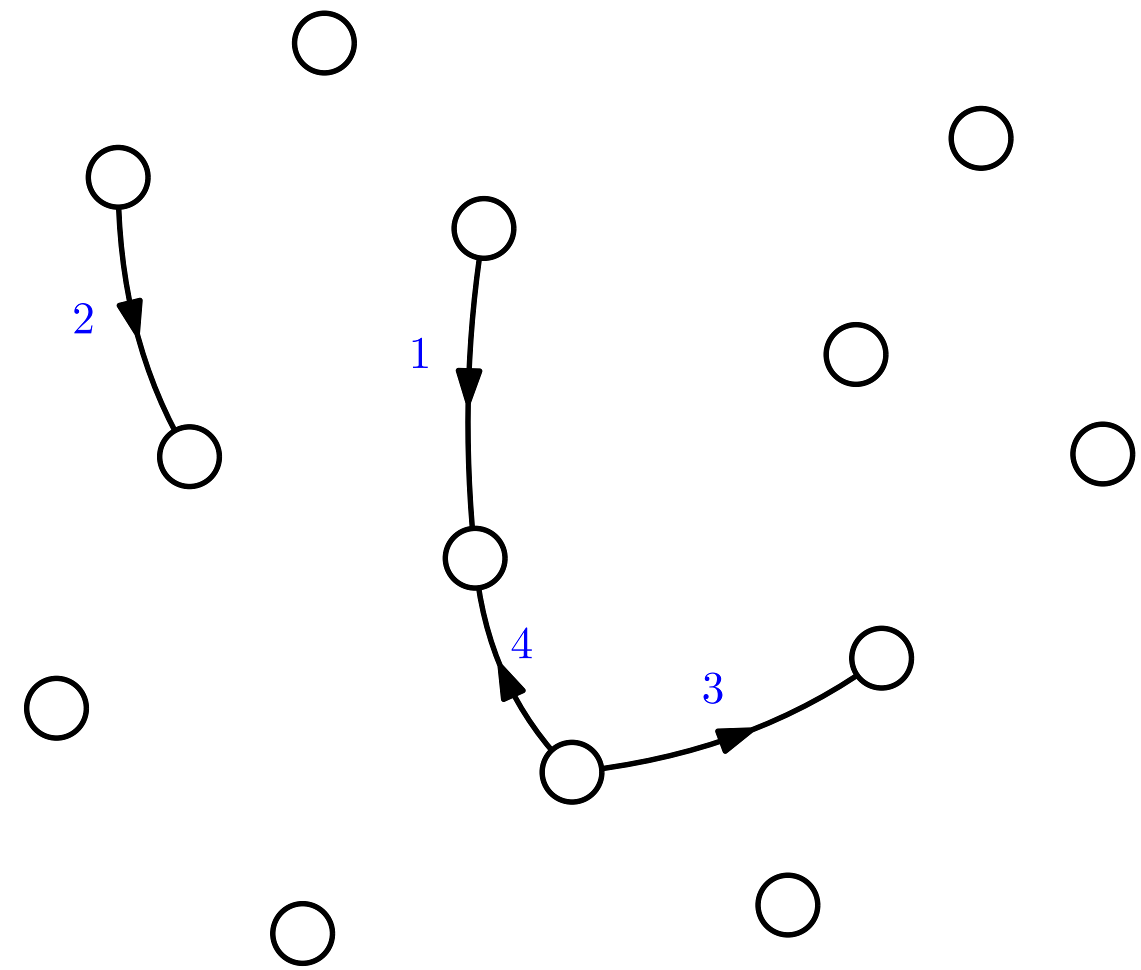

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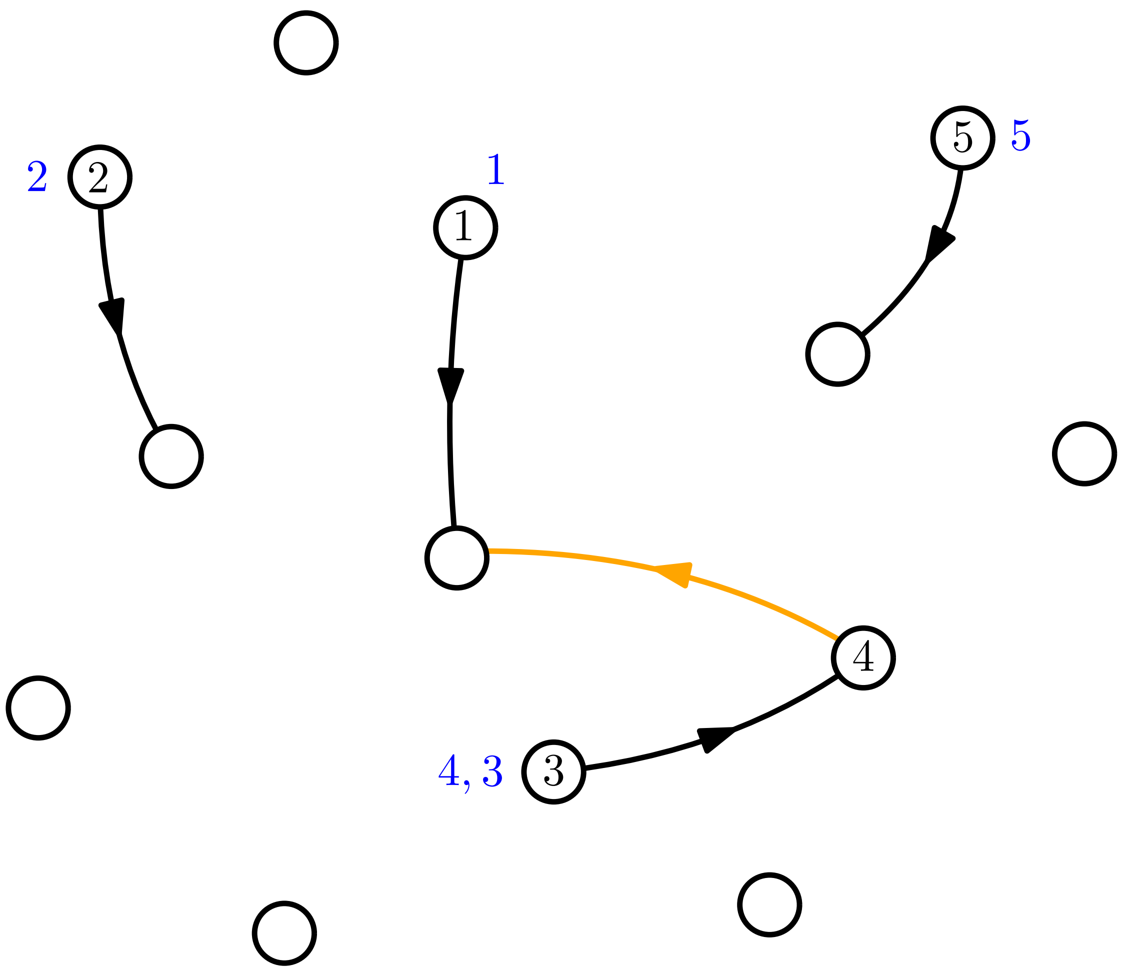

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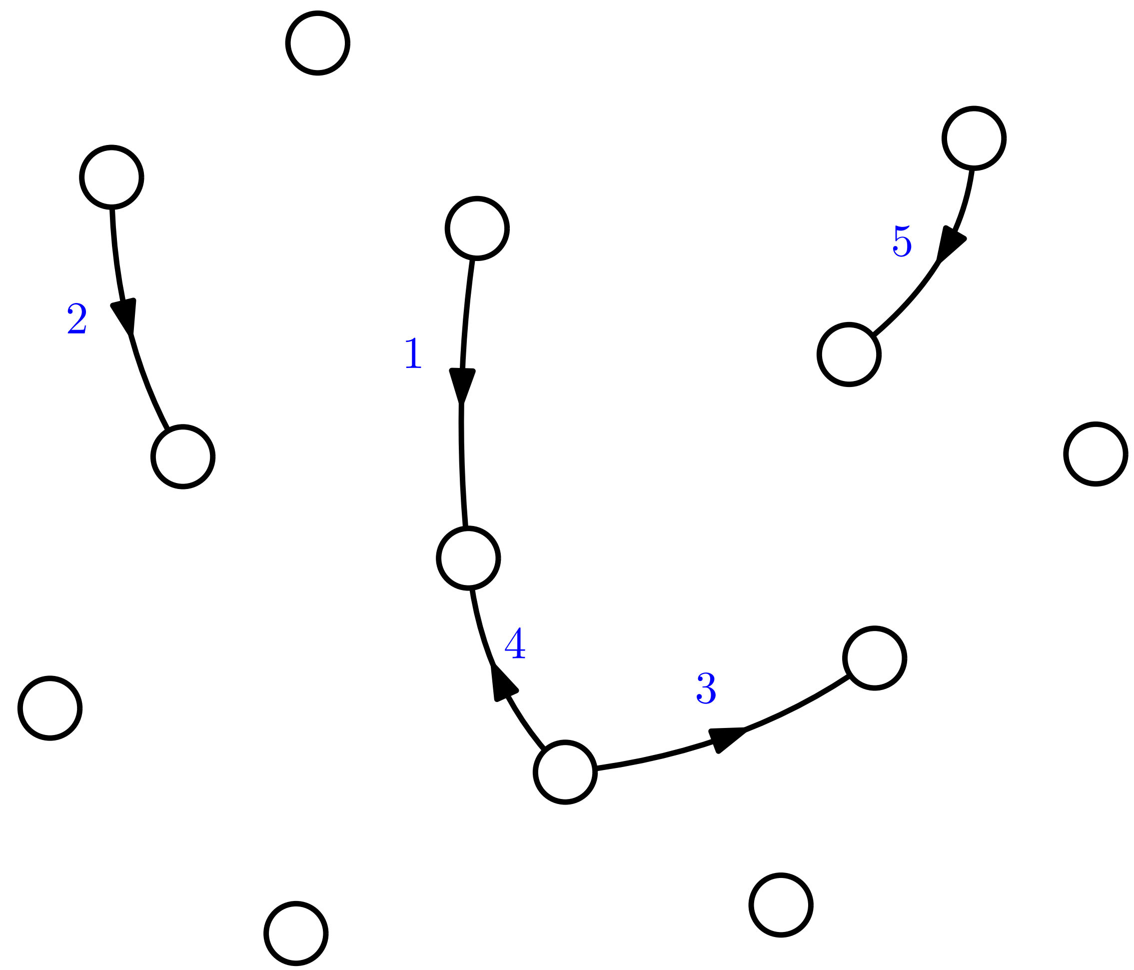

Coupling

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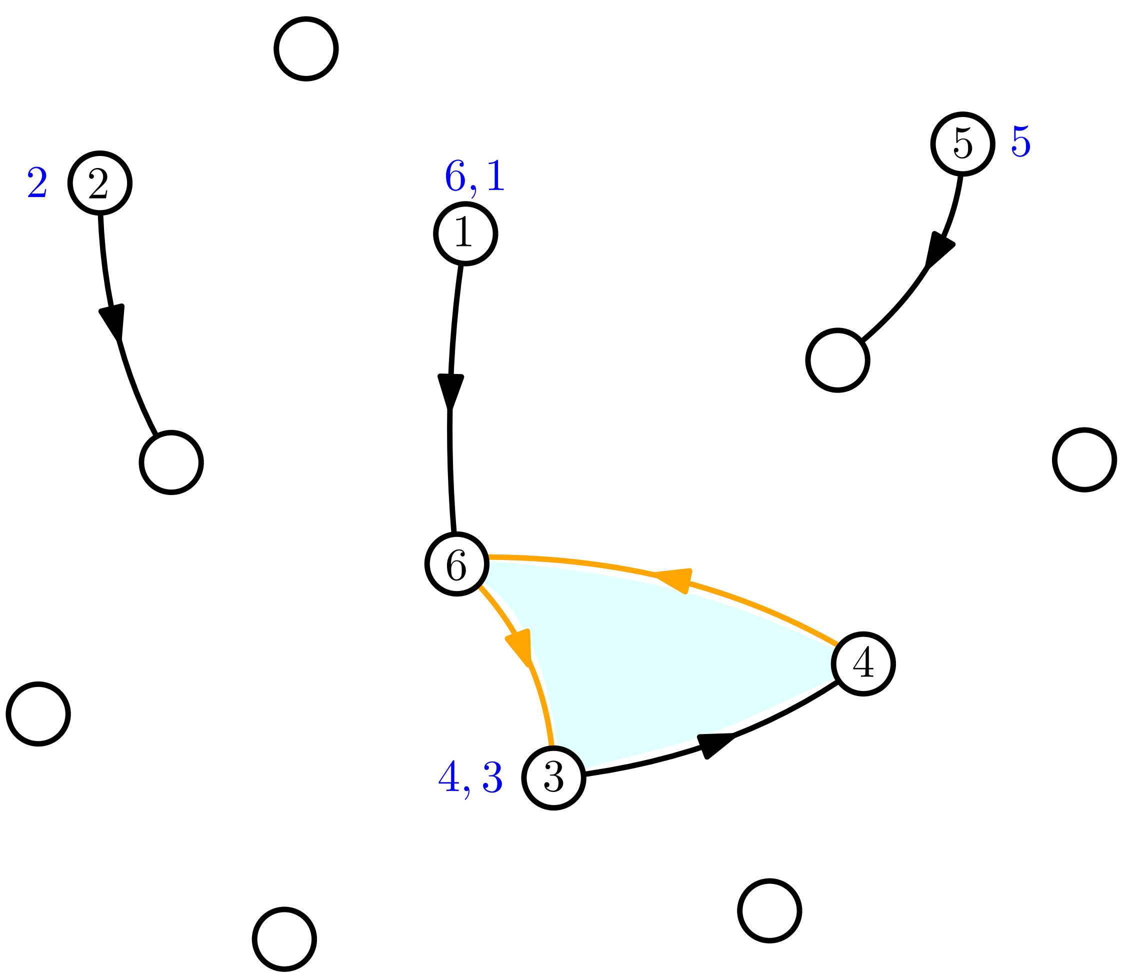

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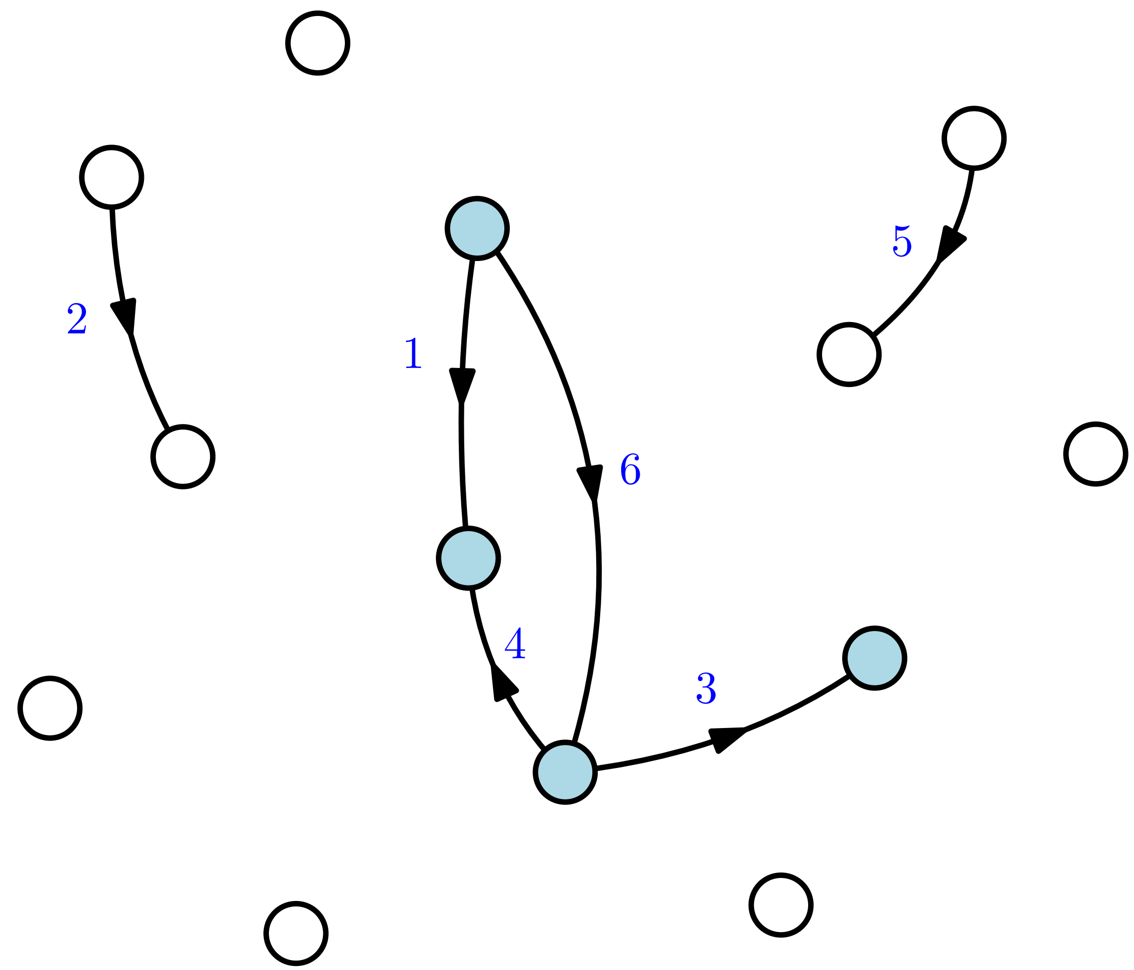

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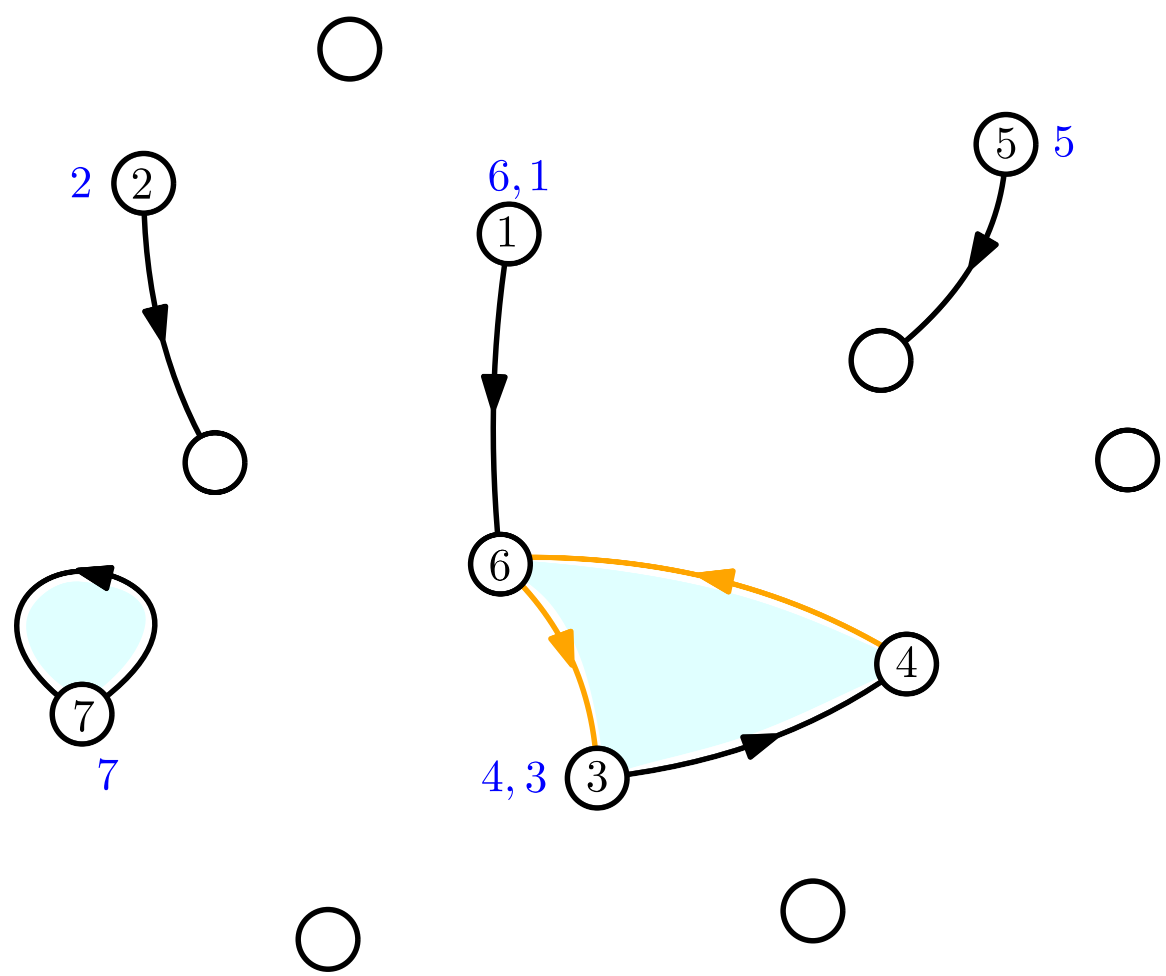

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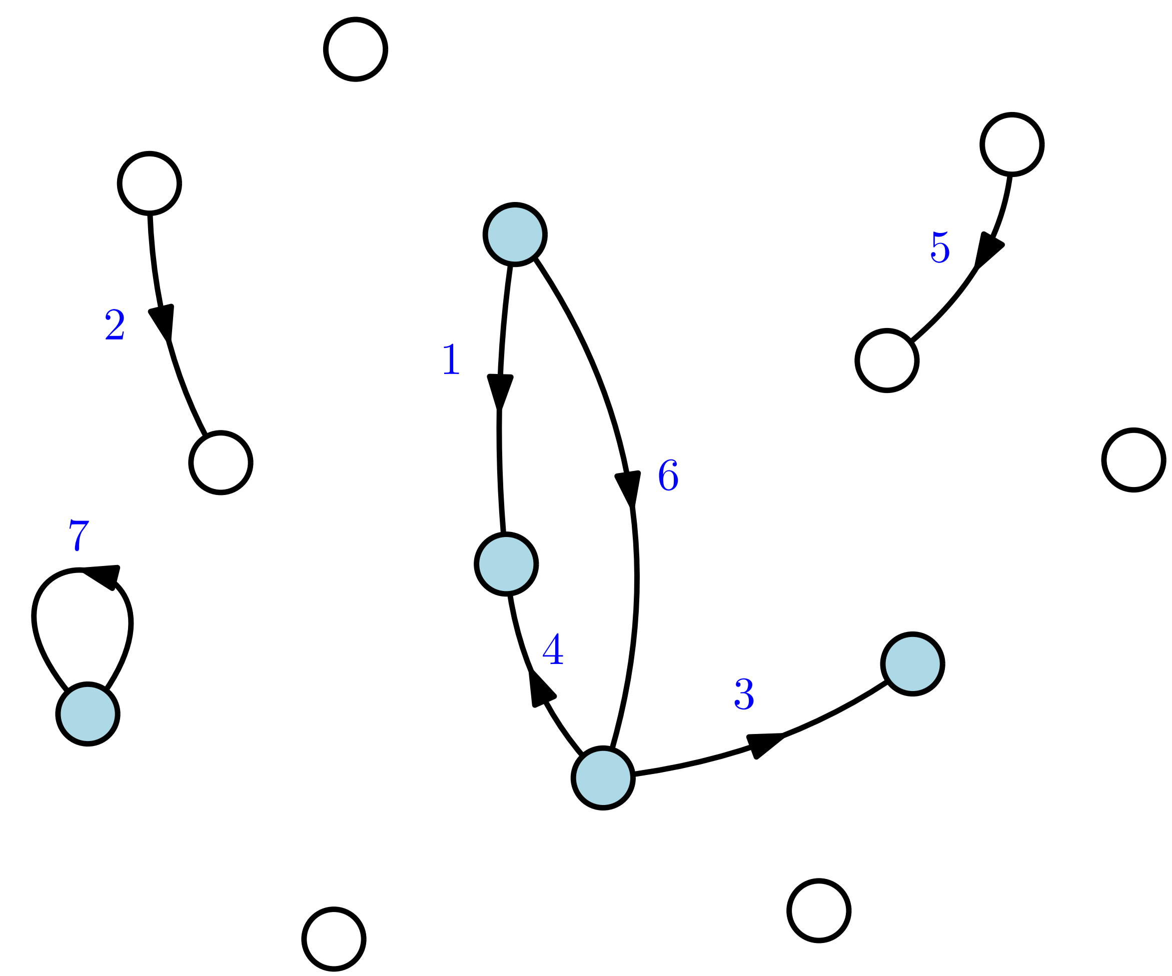

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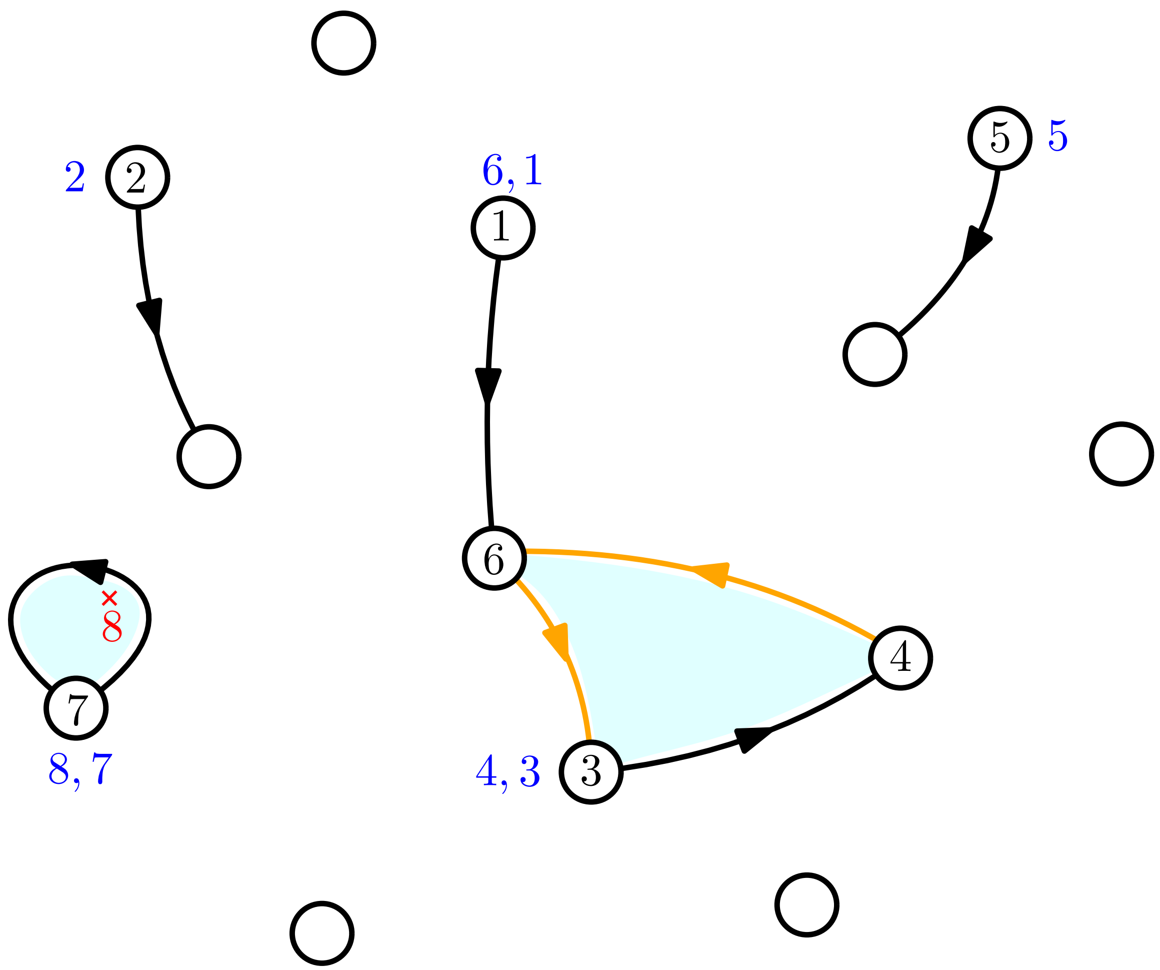

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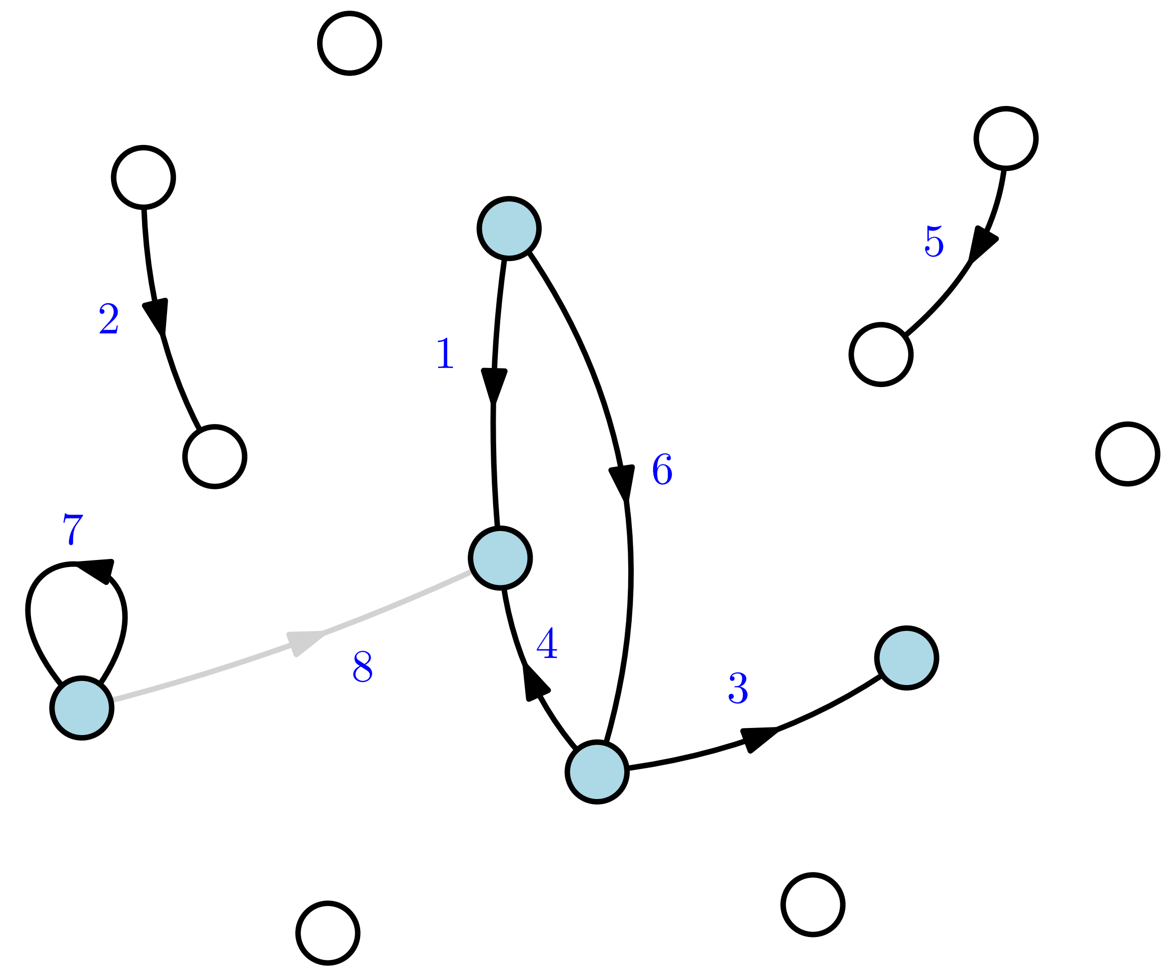

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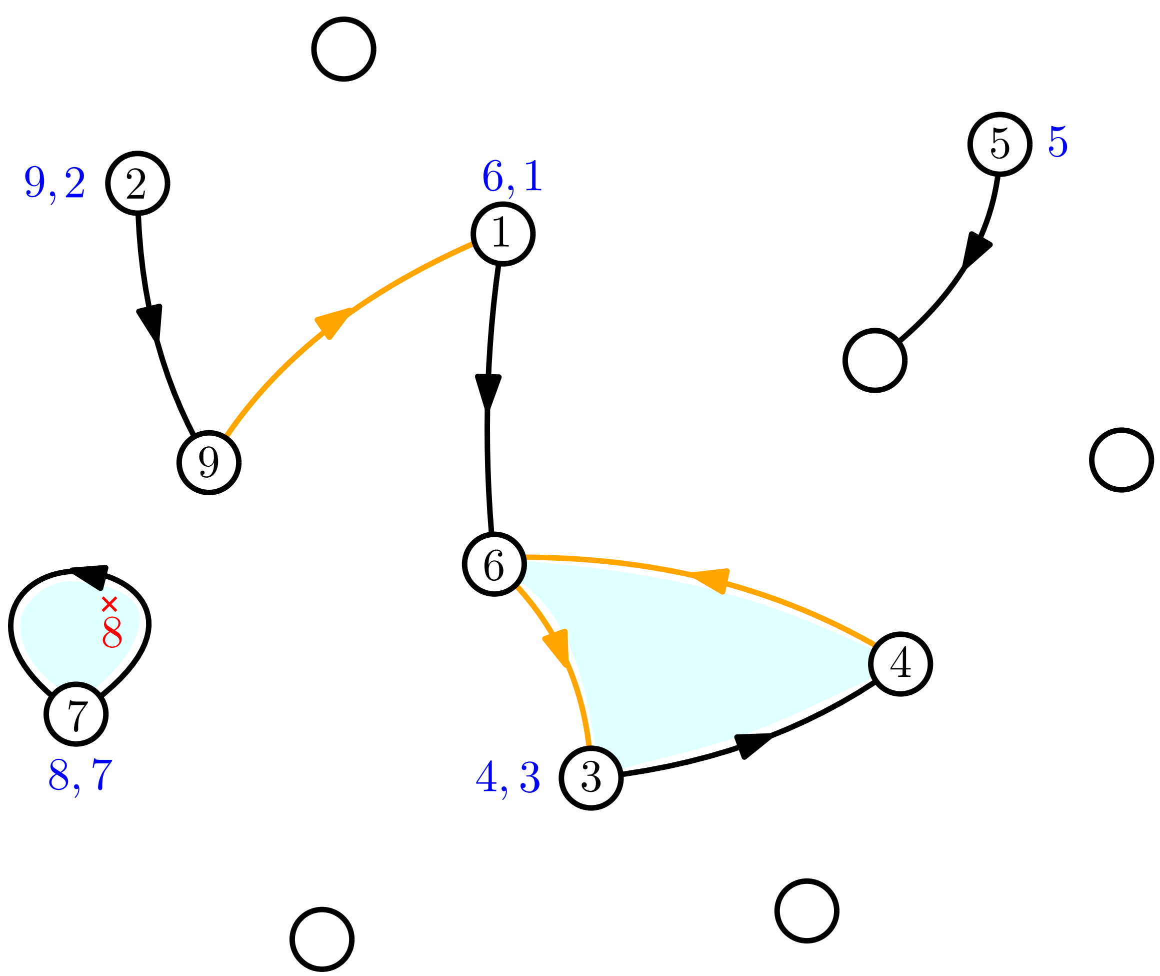

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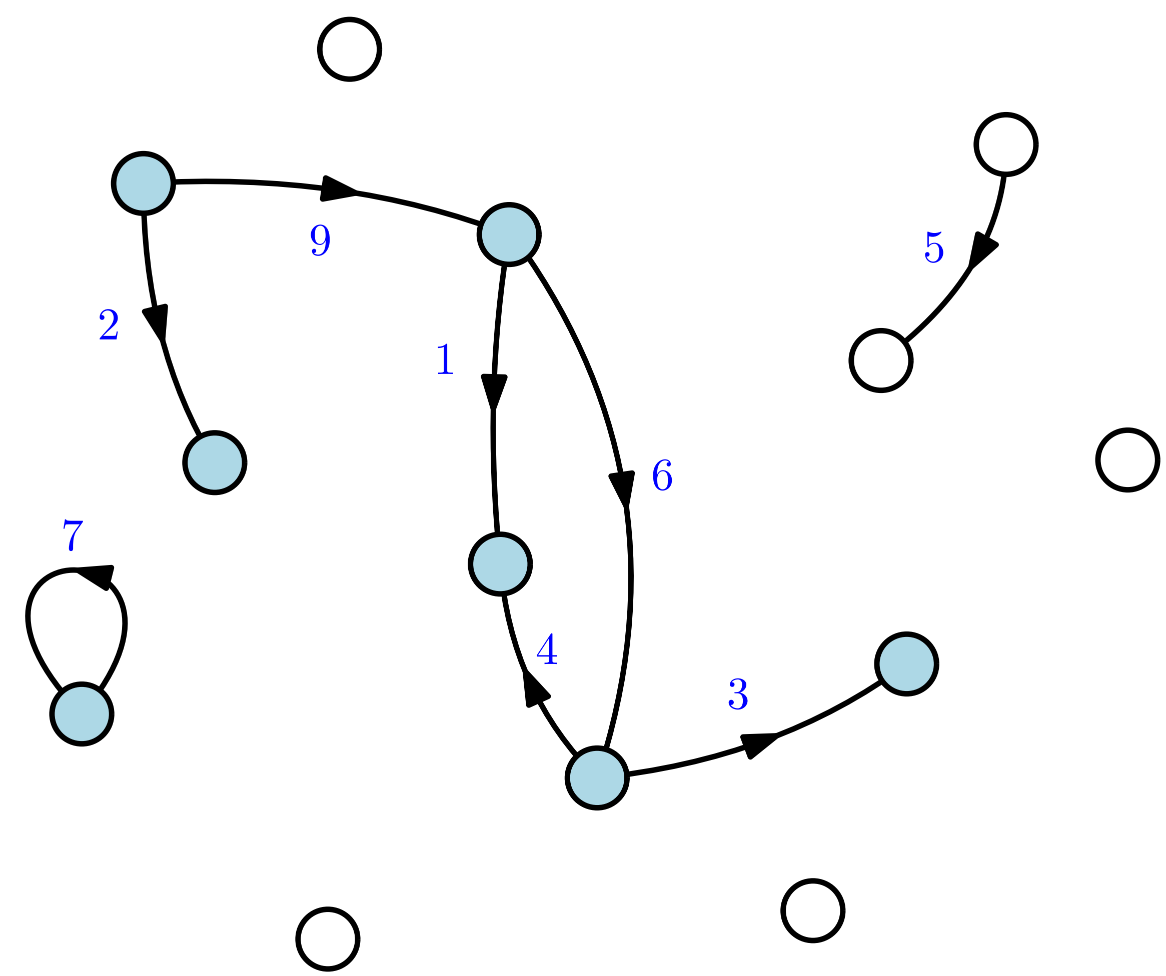
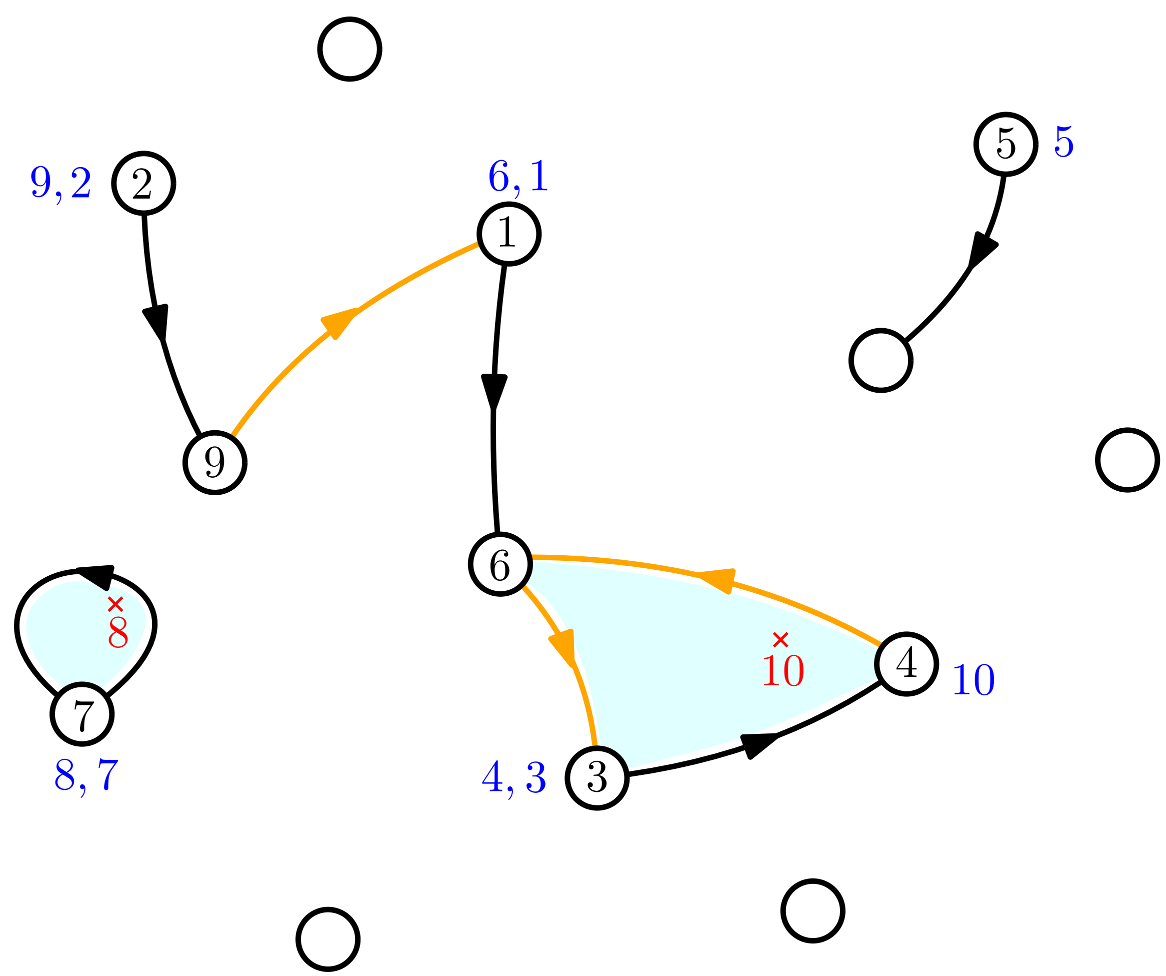


Coupling

Parking on a mapping



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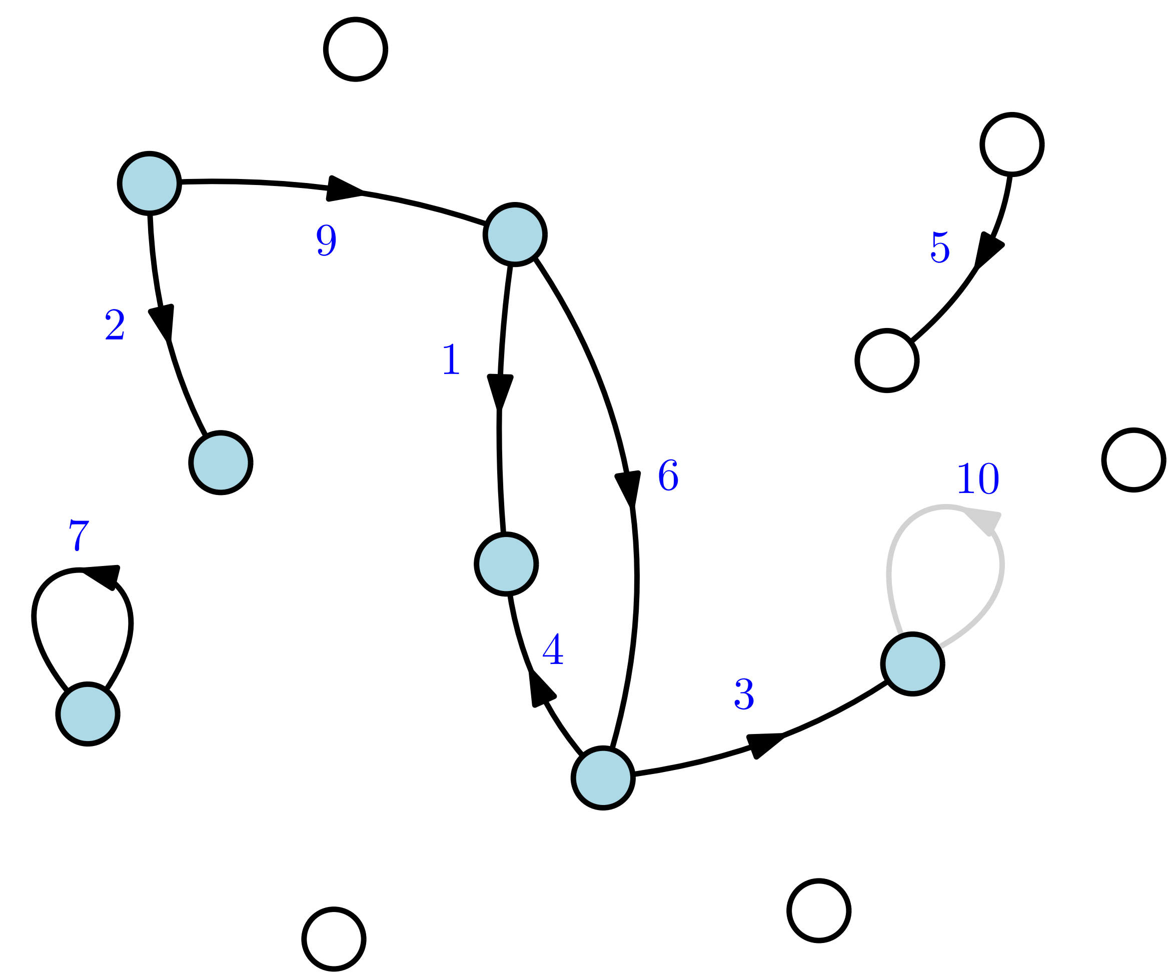

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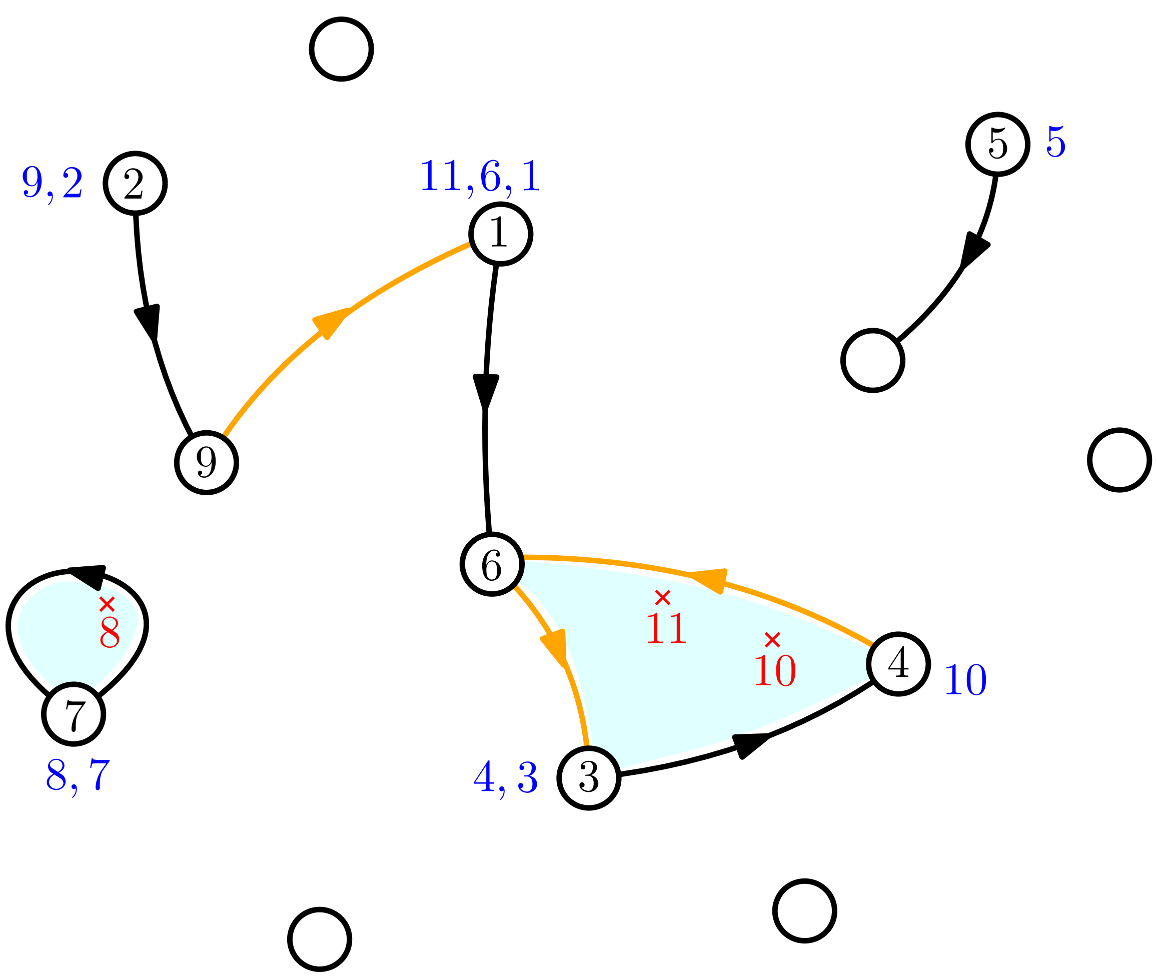

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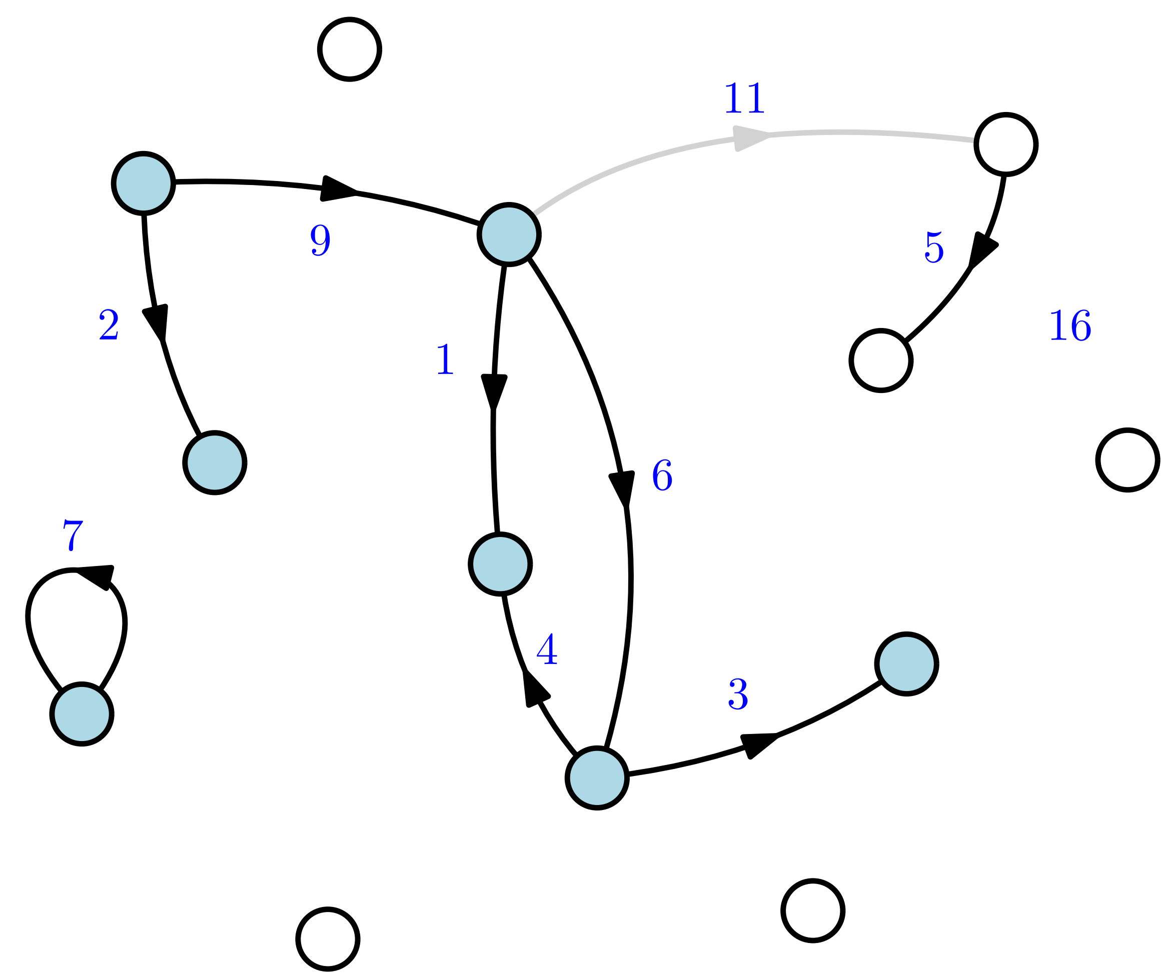

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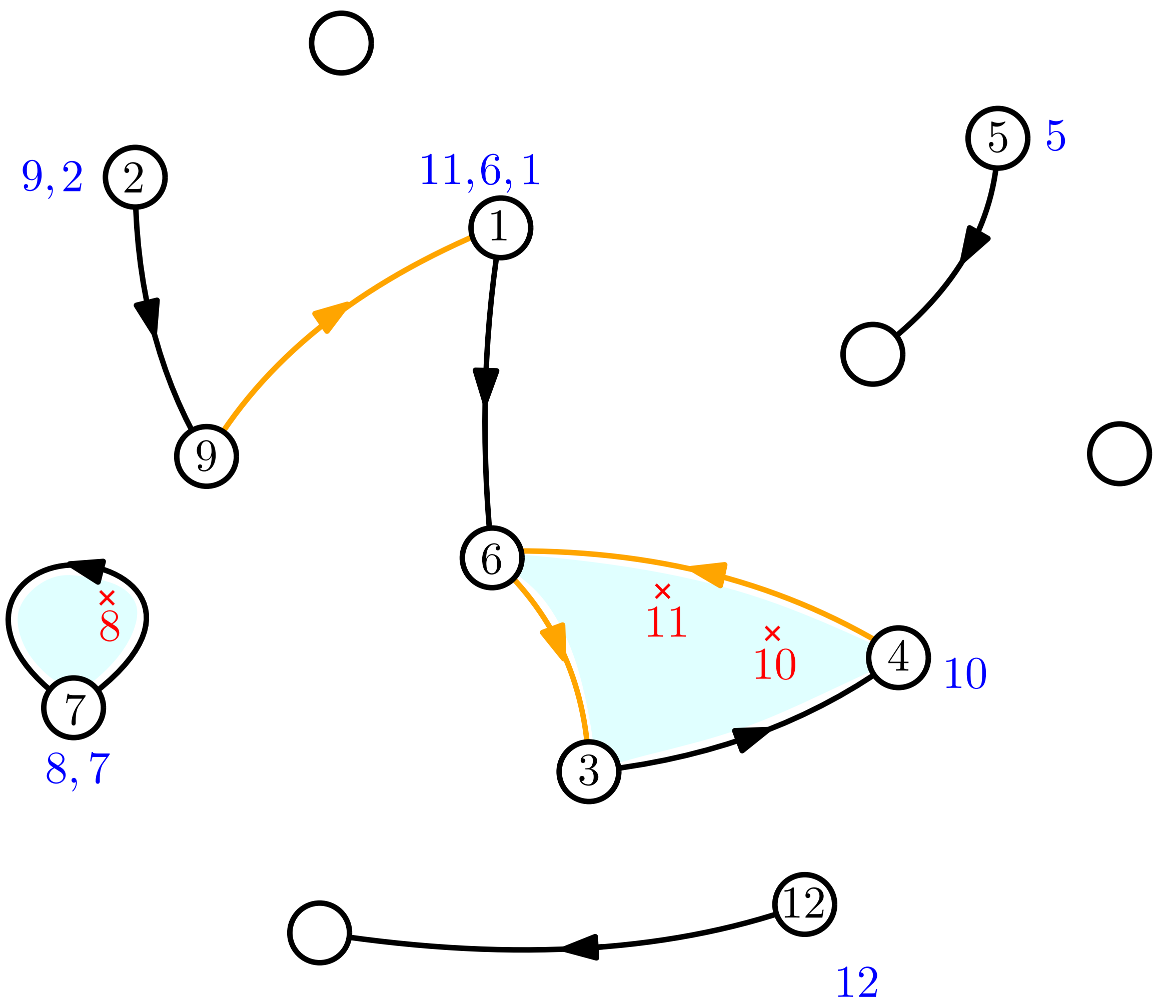

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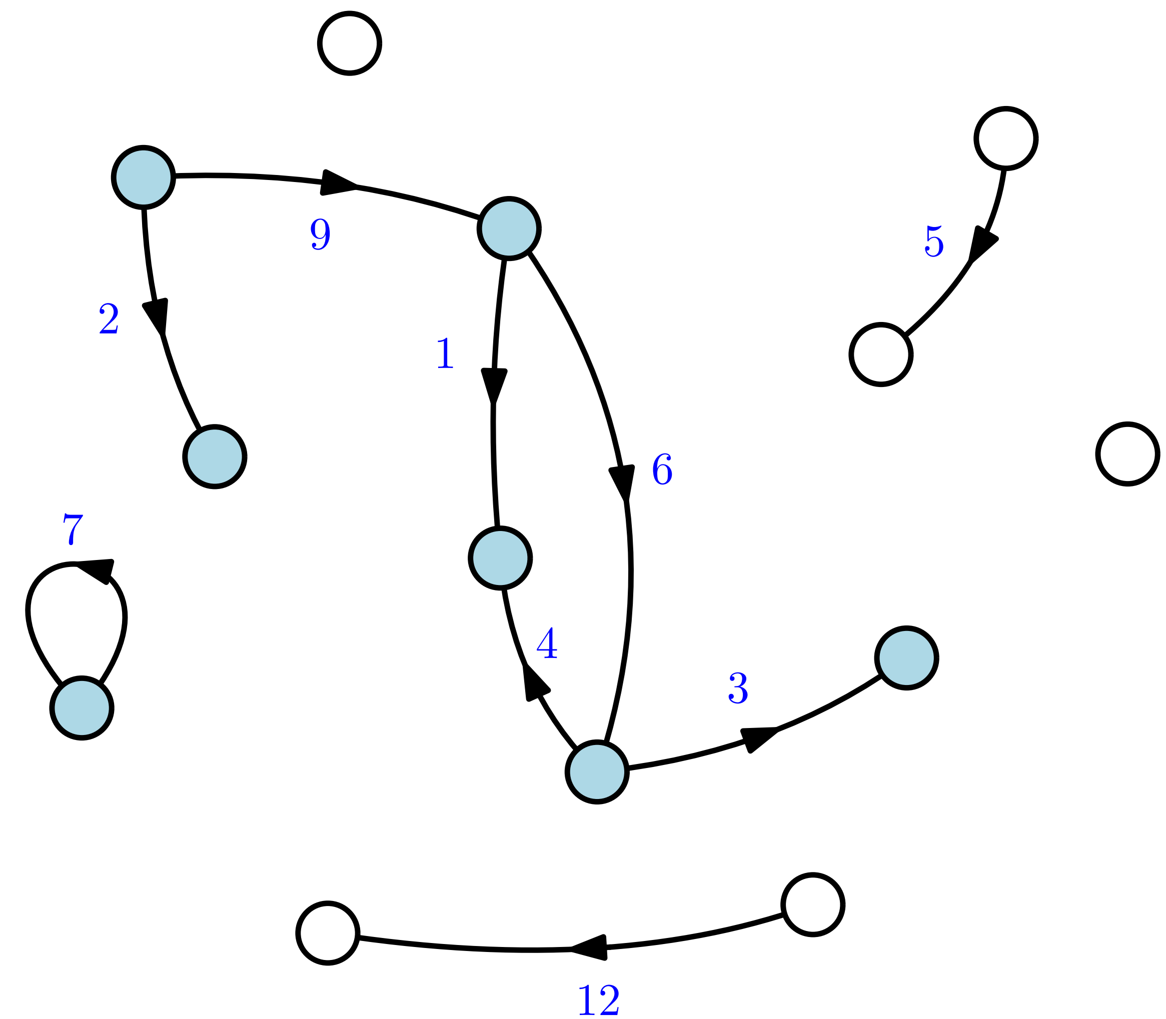

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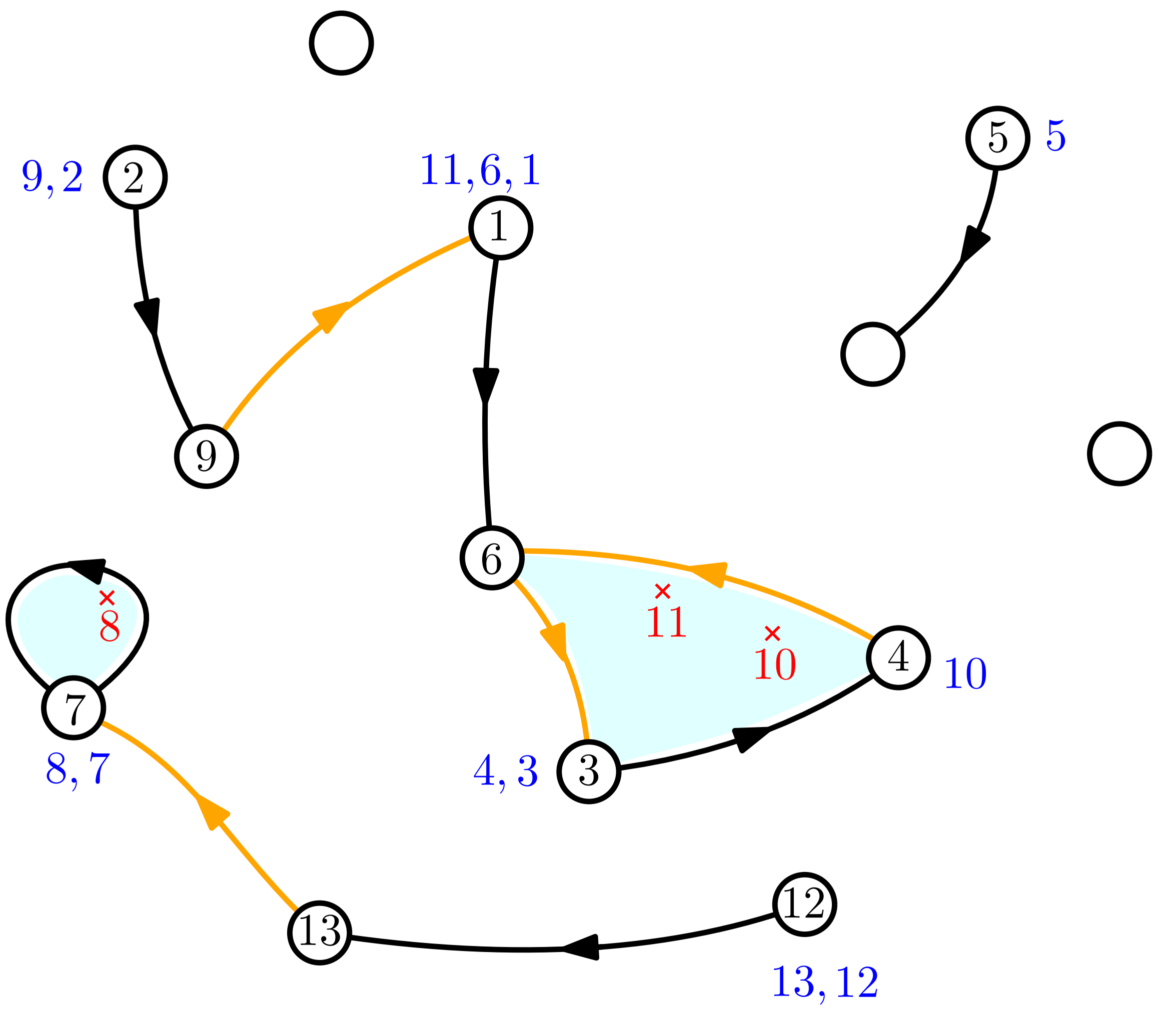

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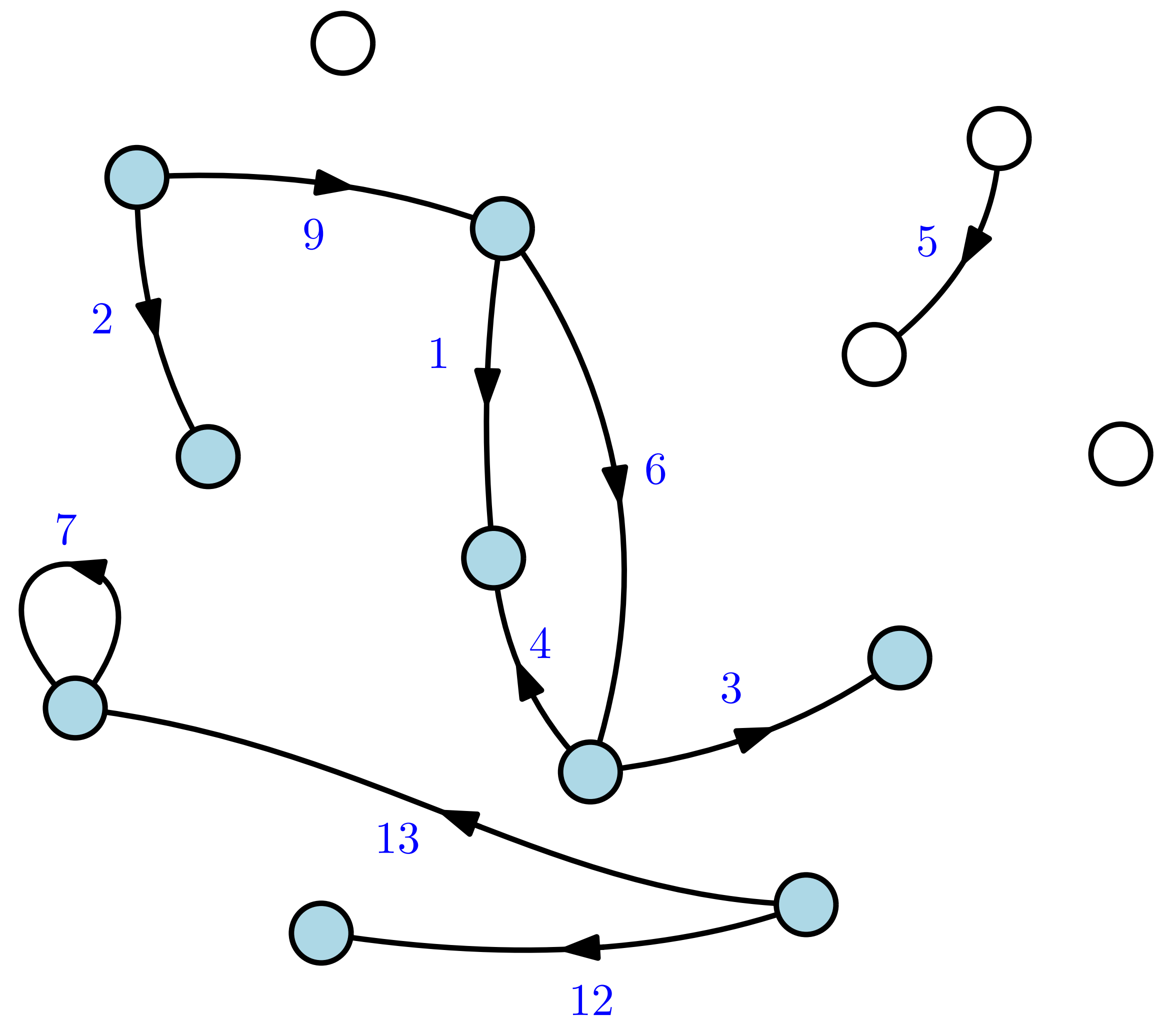

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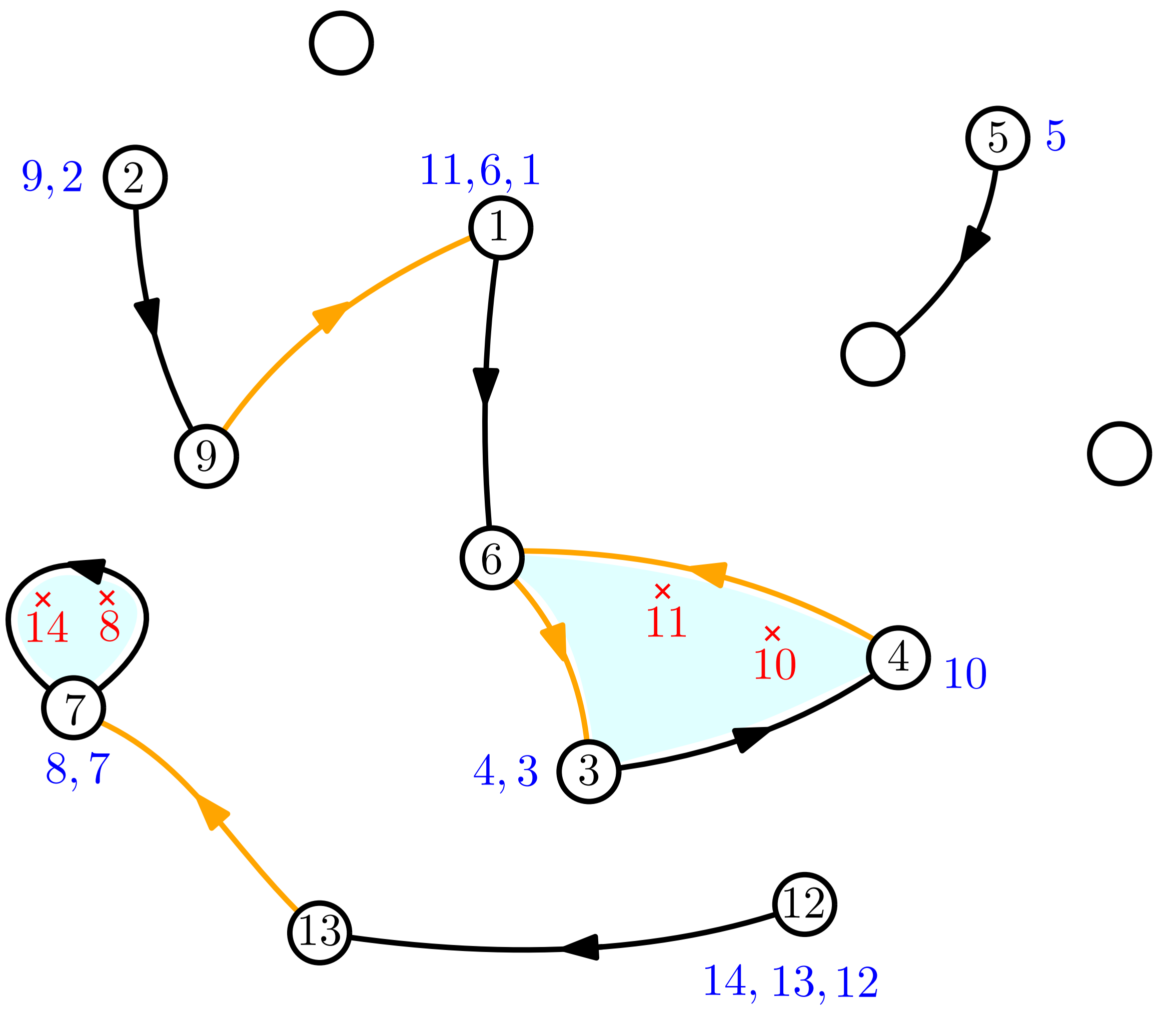

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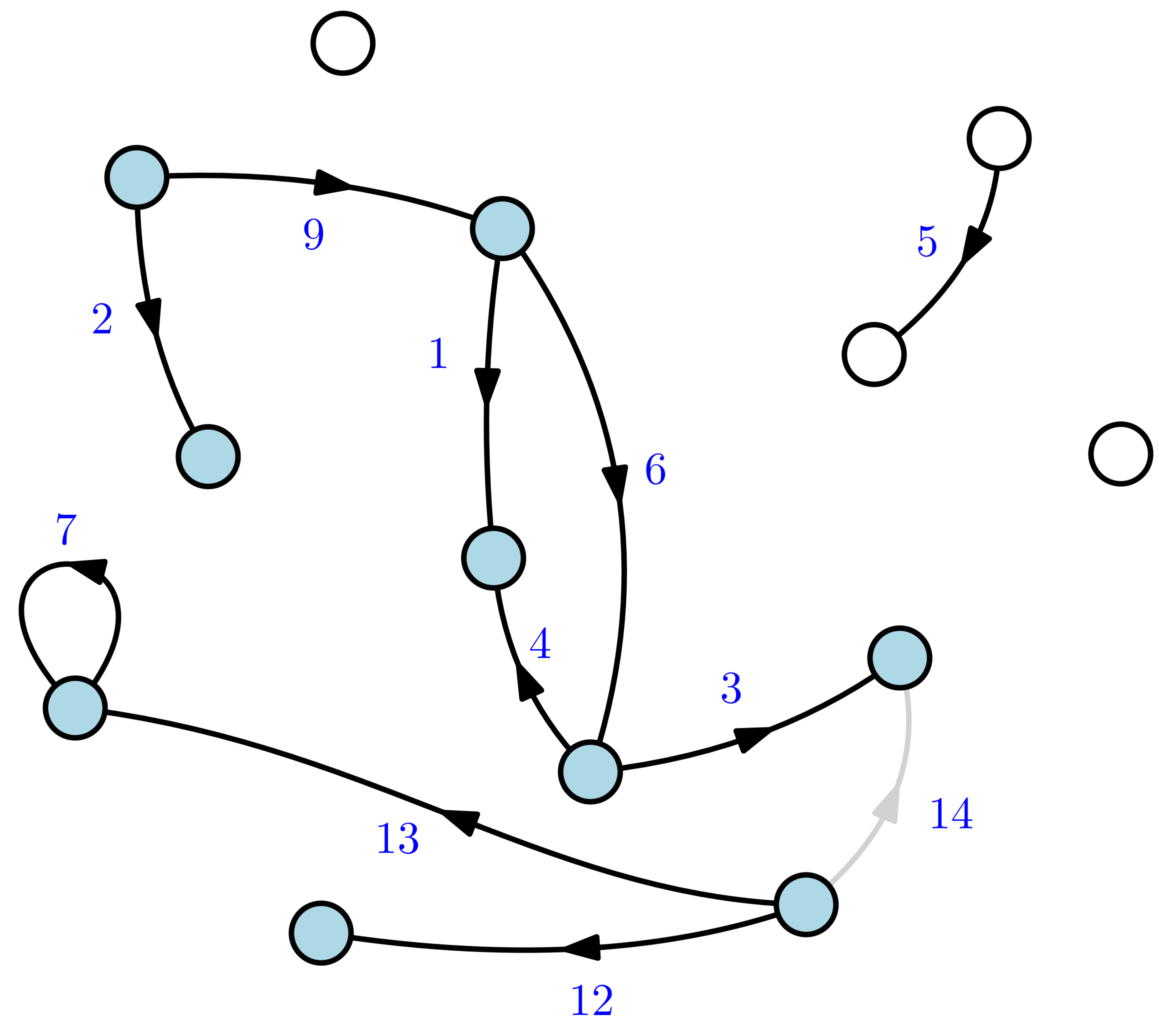

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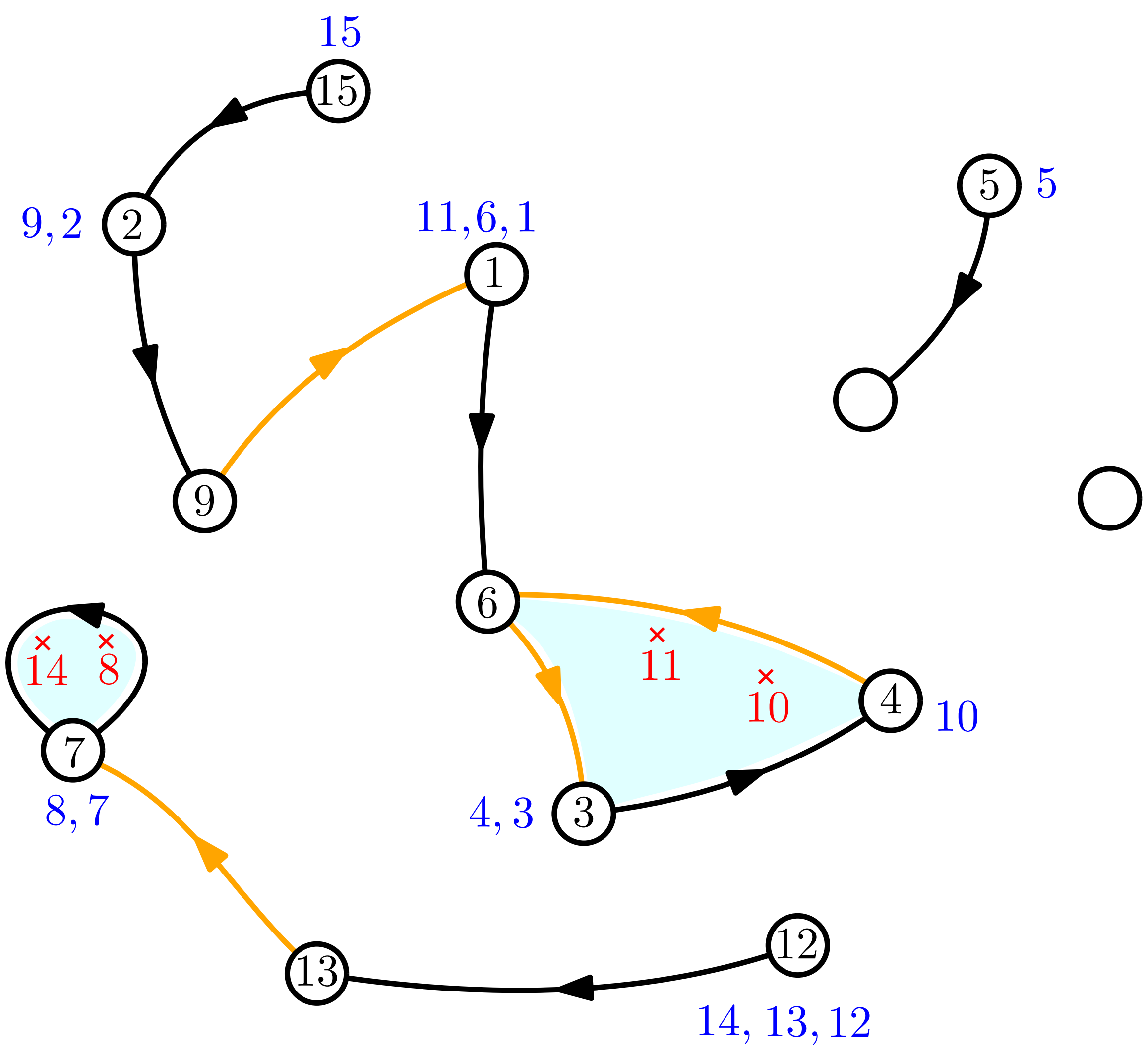

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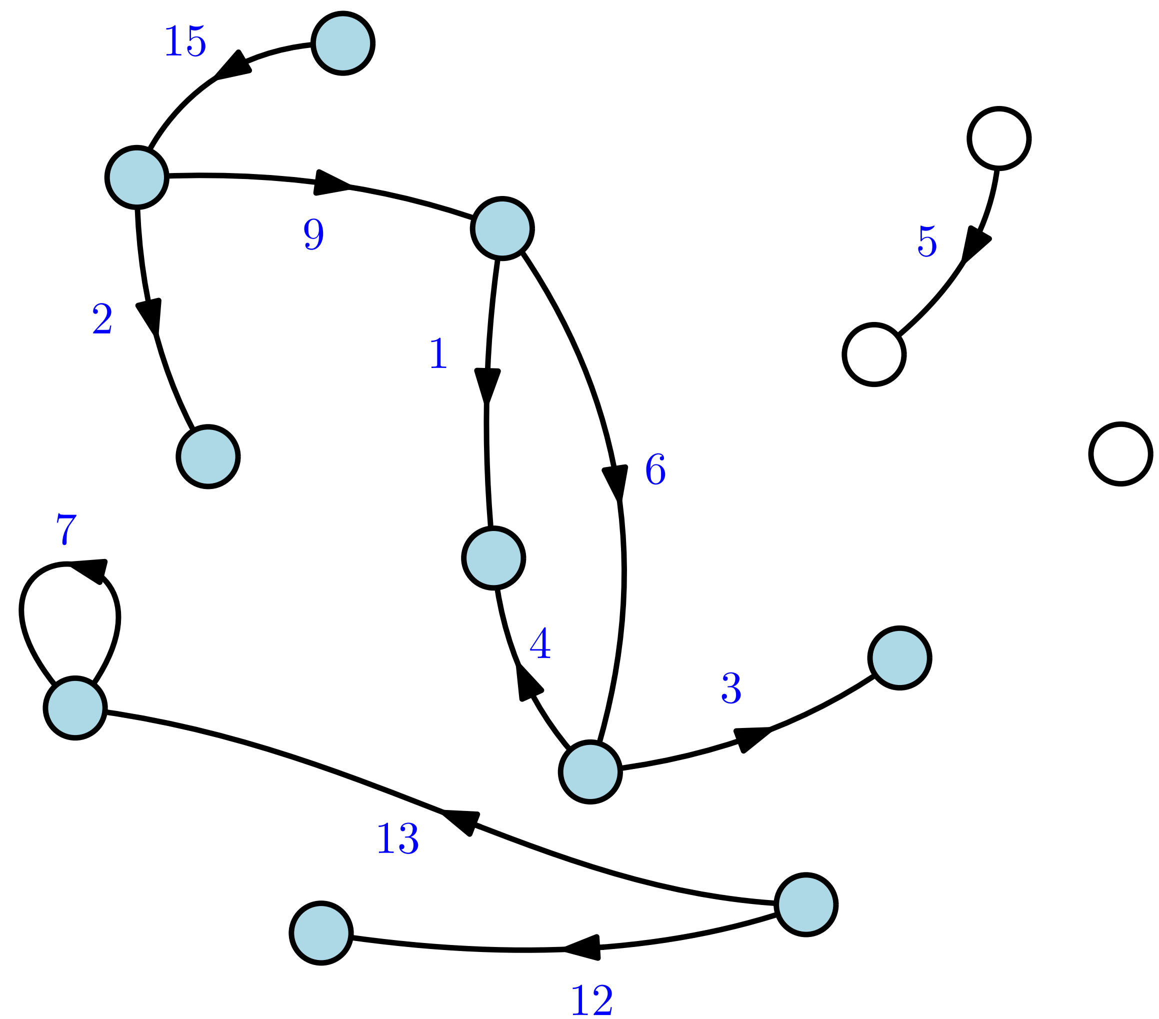
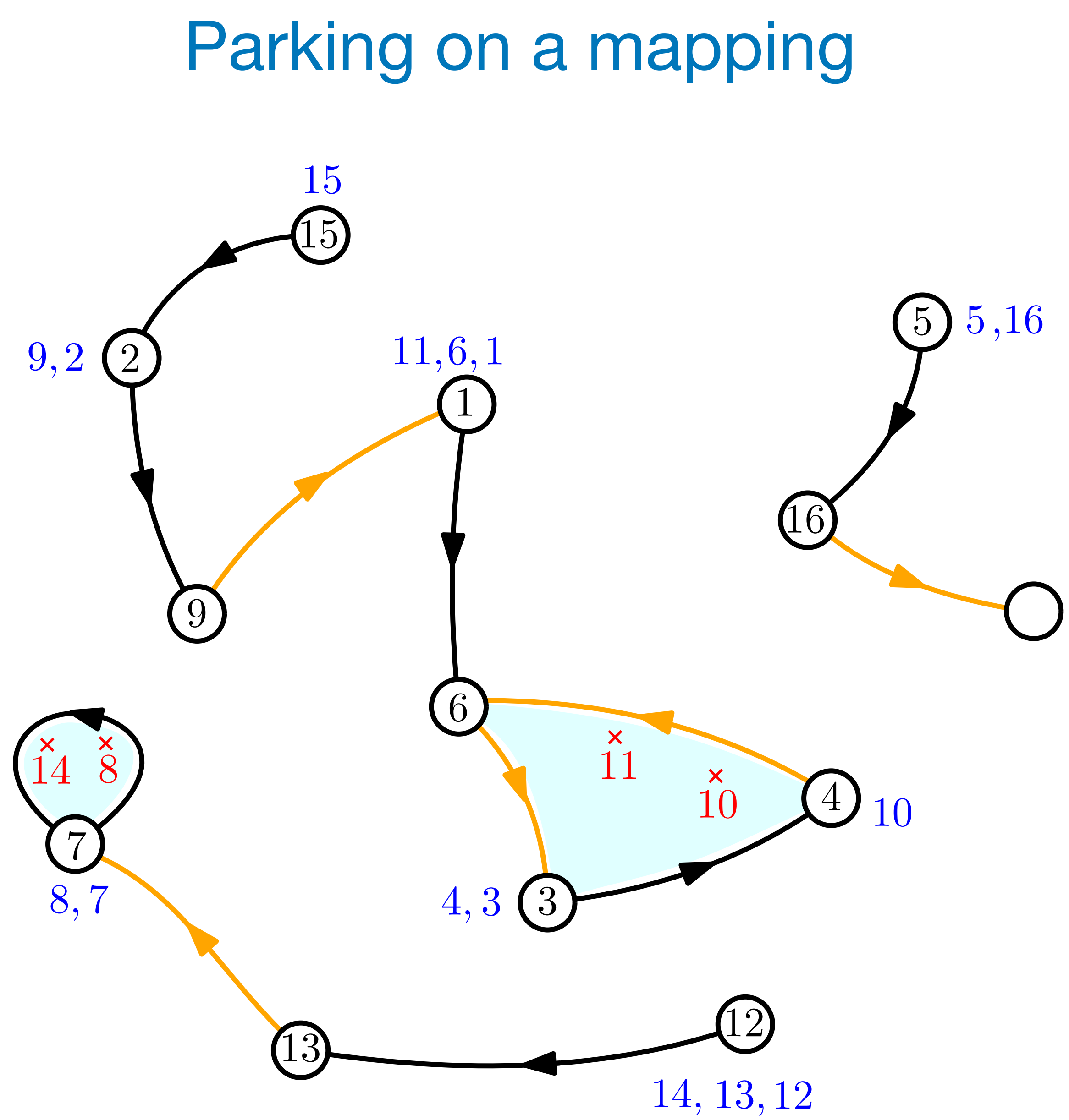
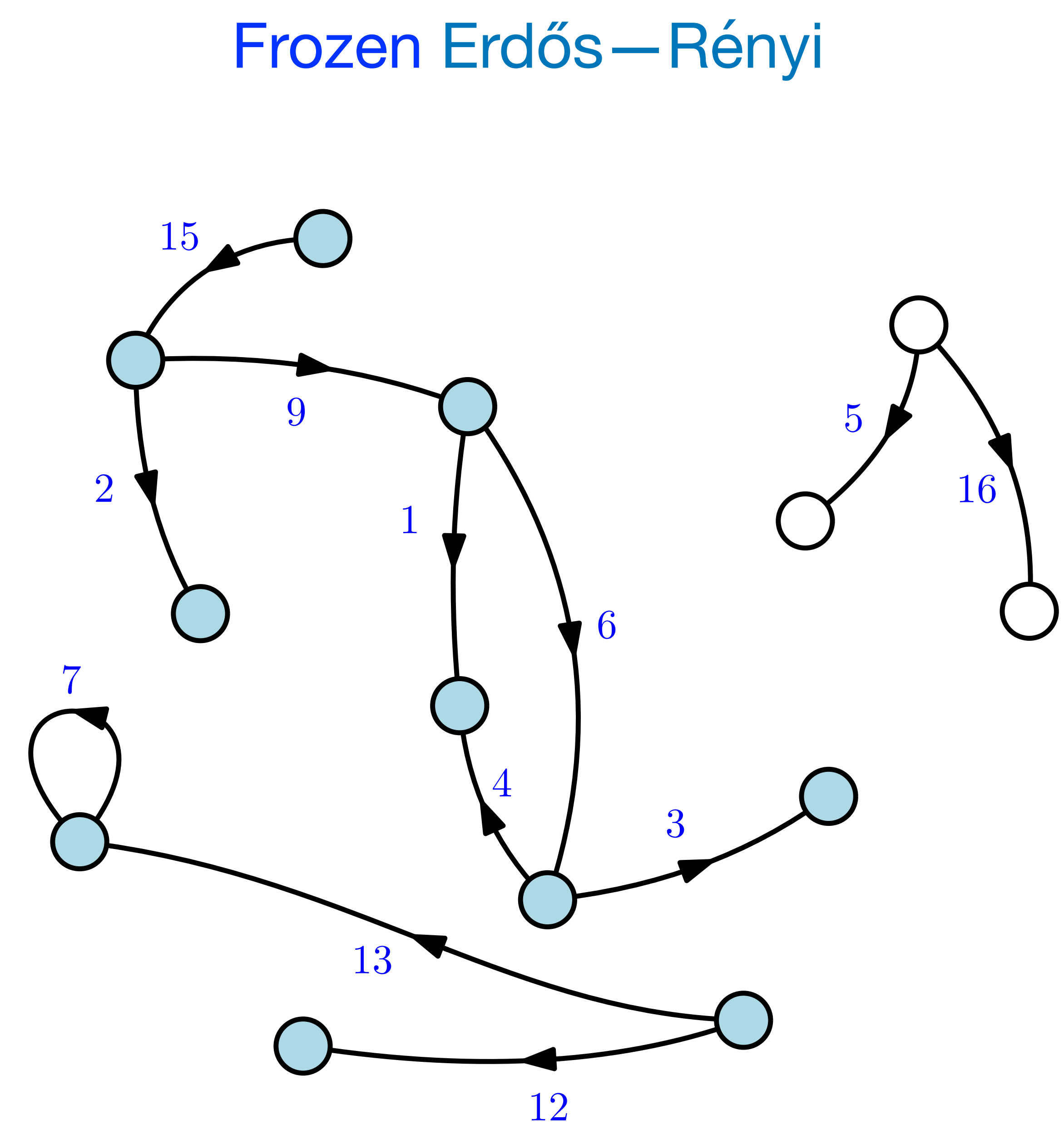


Coupling



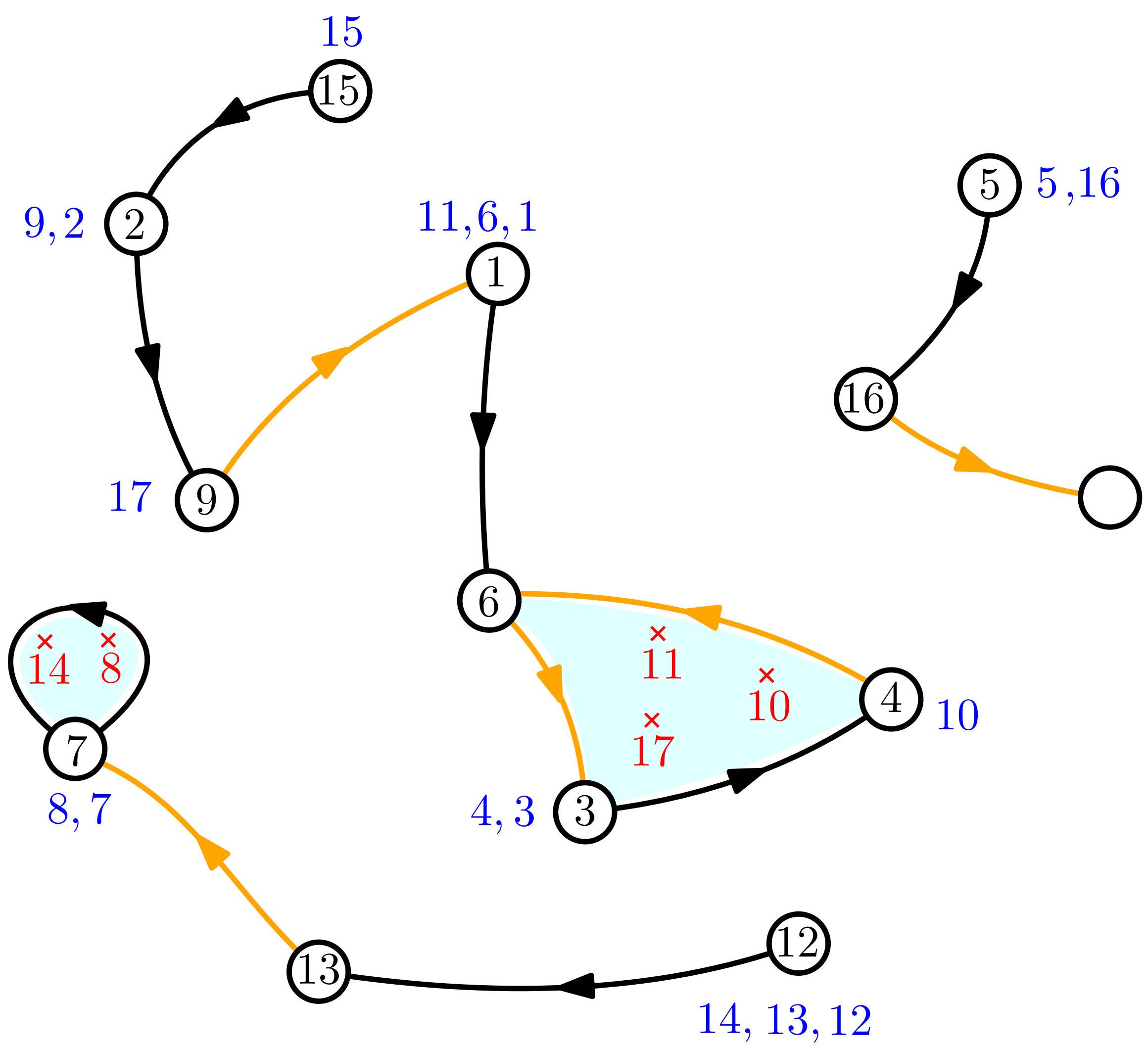
Parking on a mapping
15
15
9, 2
2
11, 6, 1
1
5
5, 16
16
9
6
14
8
11
10
7
4
10
8, 7
4, 3
3
13
12
14, 13, 12



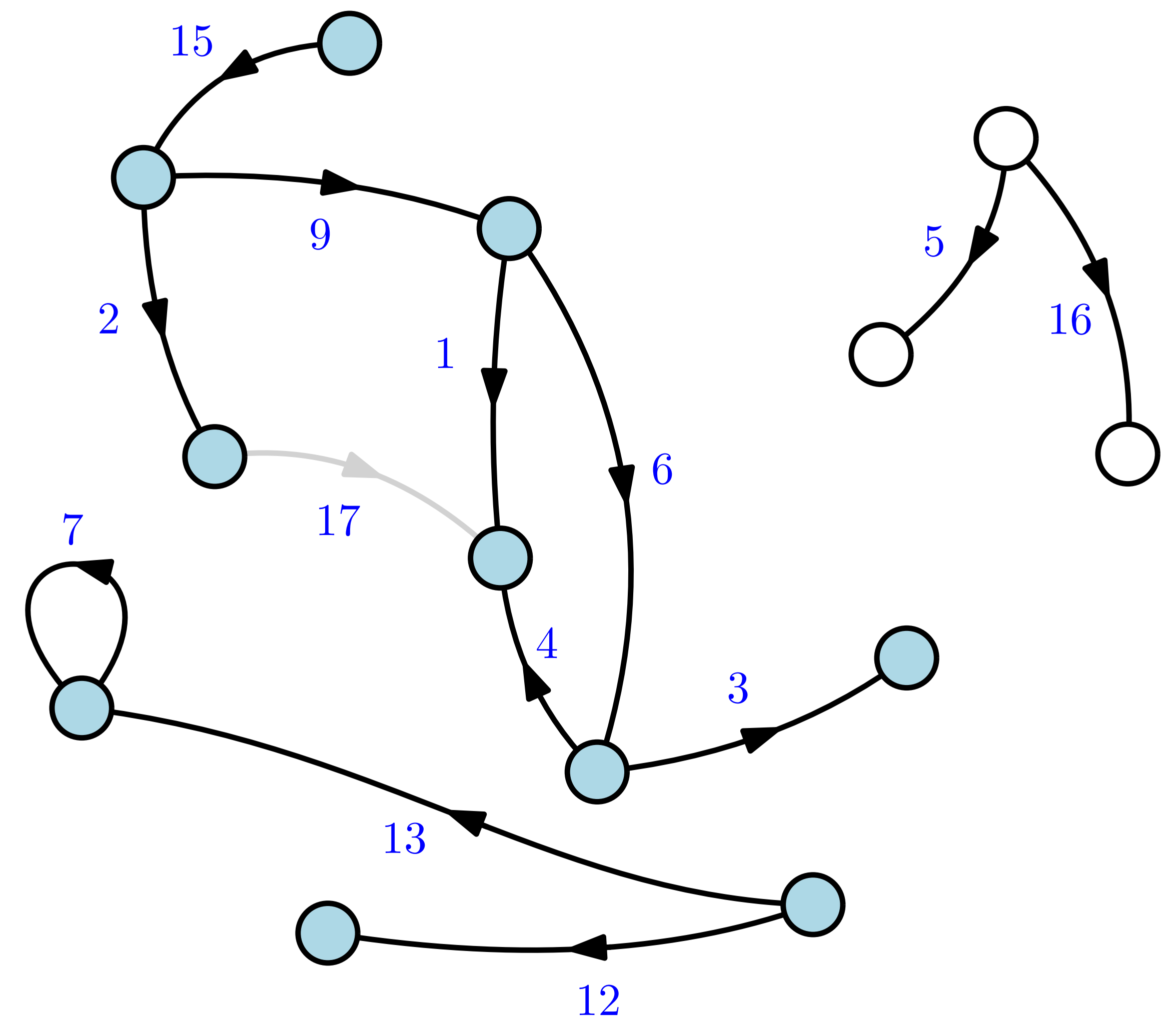
Frozen Erdős—Rényi
15
9
2
1
6
5
16
7
4
3
13
12

Coupling

Parking on a mapping

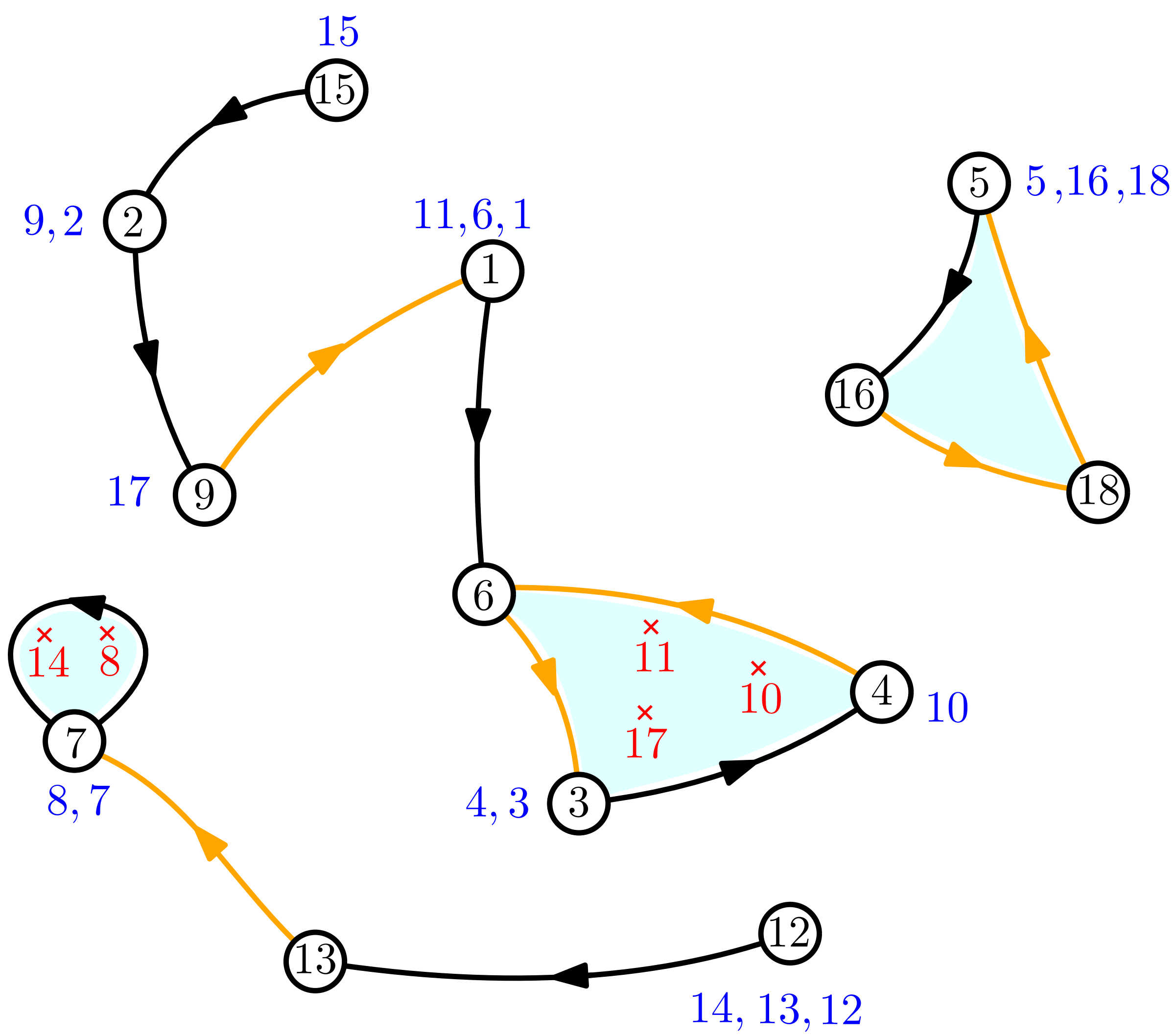


Frozen Erdős—Rényi

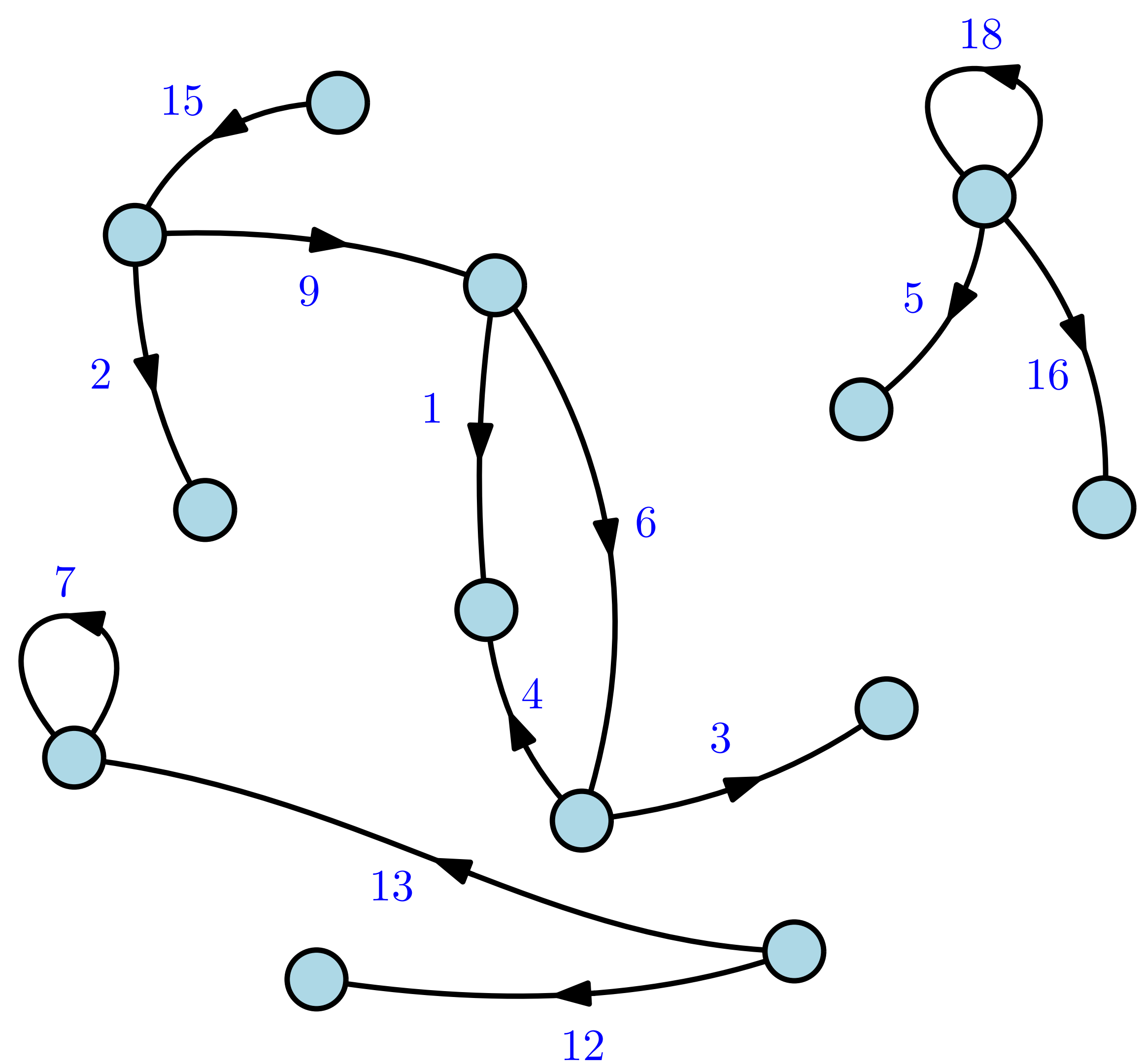


Coupling

Parking on a mapping

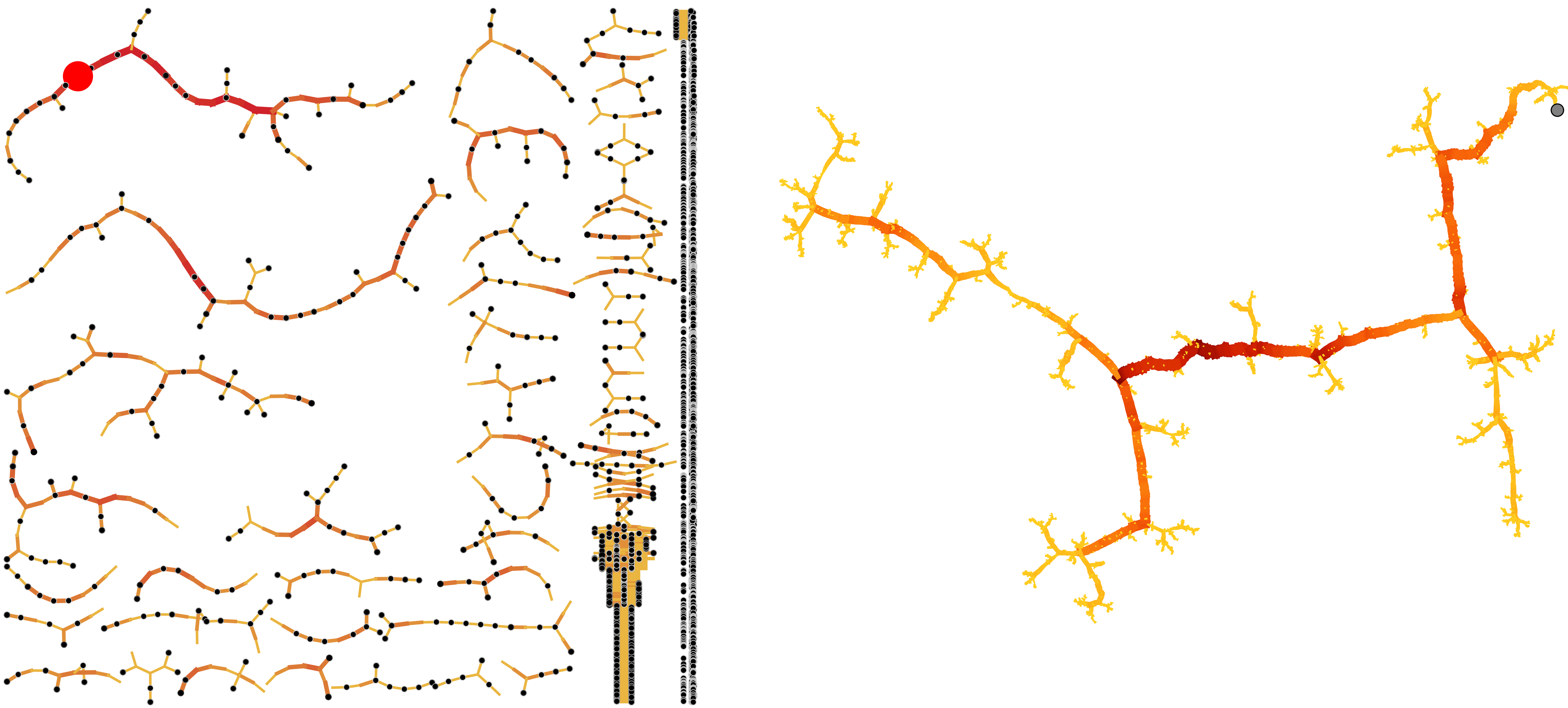


Frozen Erdős—Rényi



Consequences

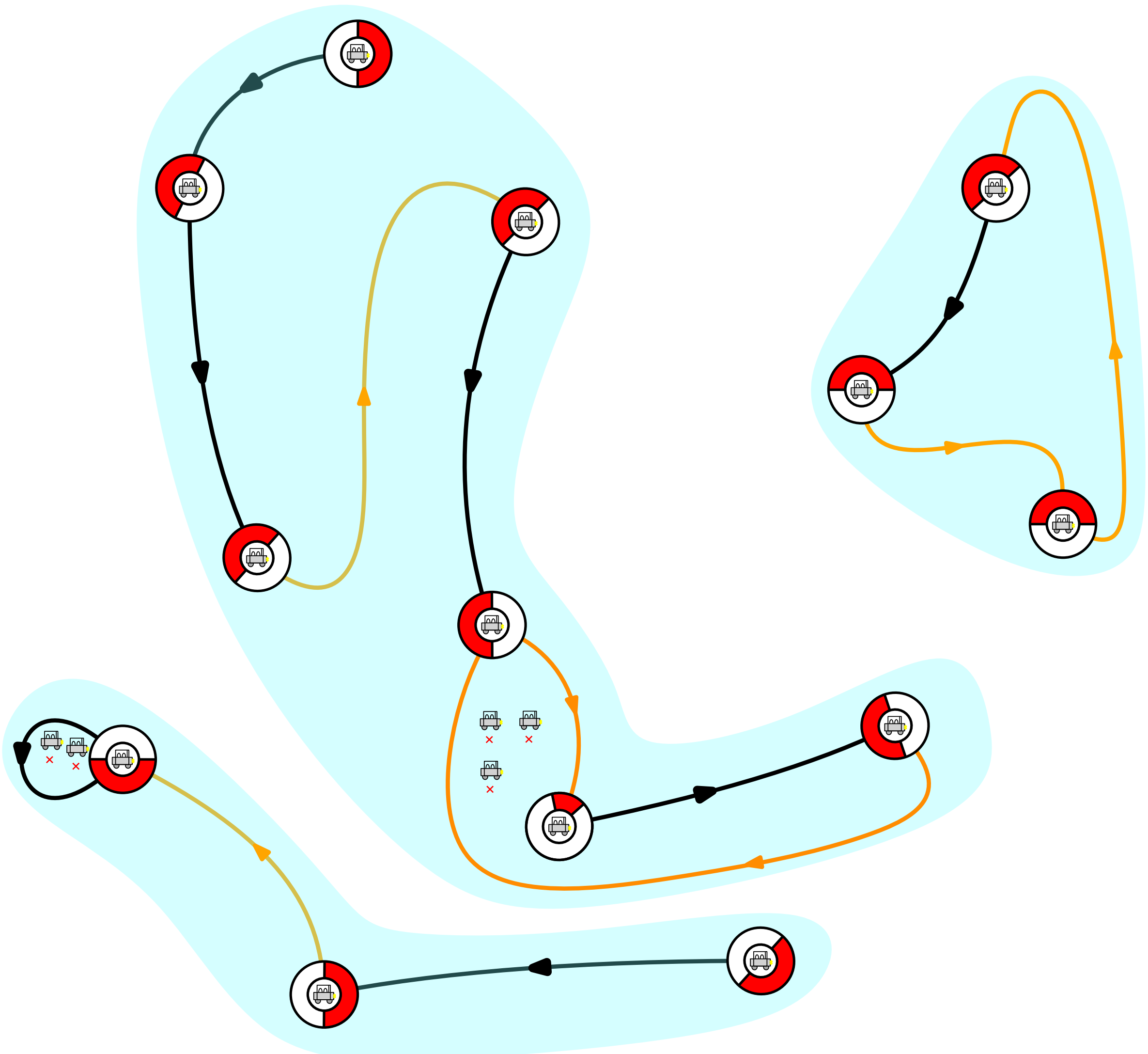
- The size of the components are the same (but not their geometry).

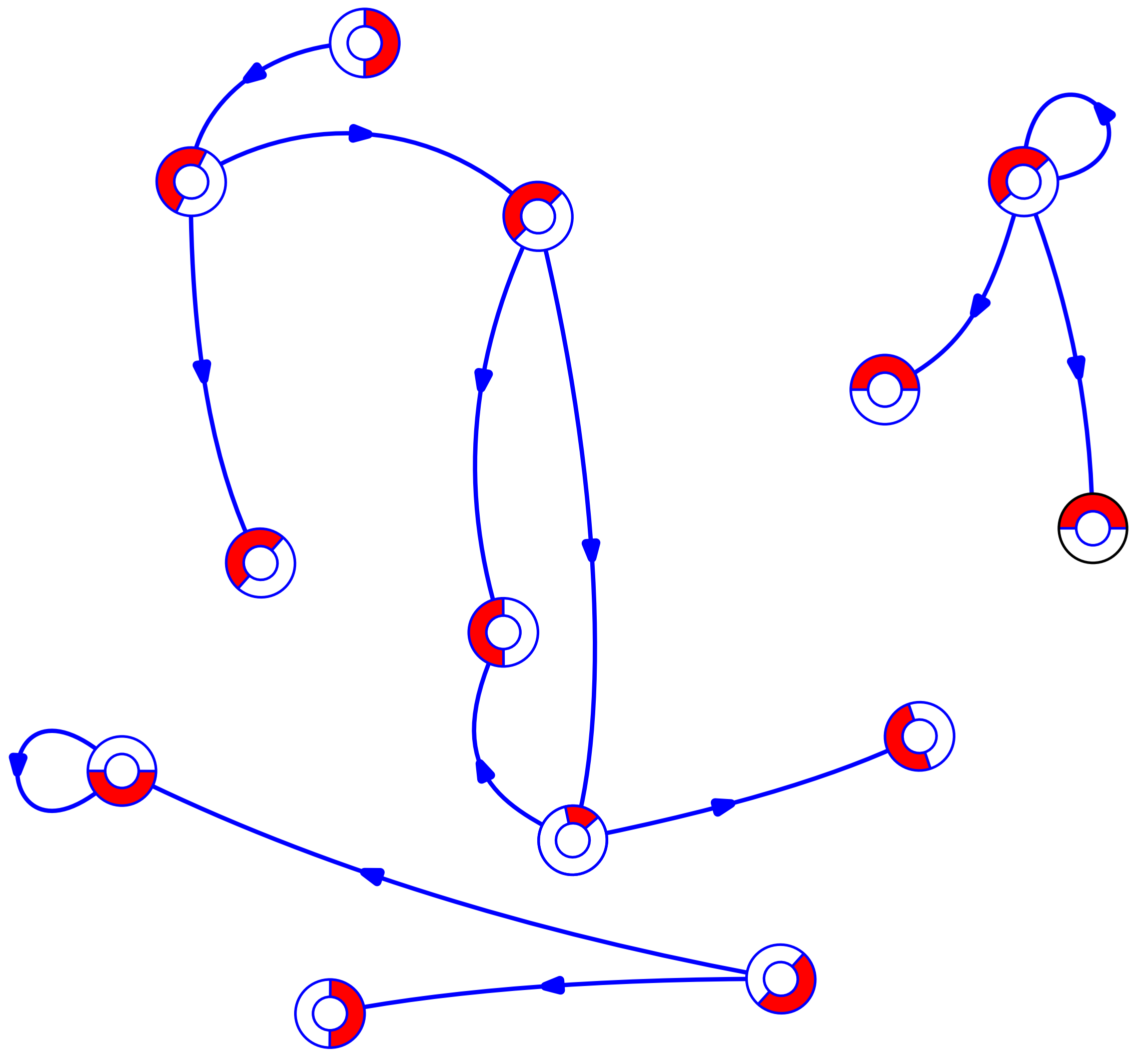


- $\mathbb{P}\left(\frac{n}{2} \text{ cars park on } T_n\right) \propto \mathbb{P}\left(G(n,\frac{n}{2}) \text{ has no cycle}\right) \sim \mathrm{cst} \cdot n^{-1/6}.$

Consequences

- Similar coupling for general Bienaymé—Galton—Watson trees and general car arrivals
- "Probabilistic" explanation of the phase transition
- Universality results





Thank you for your attention