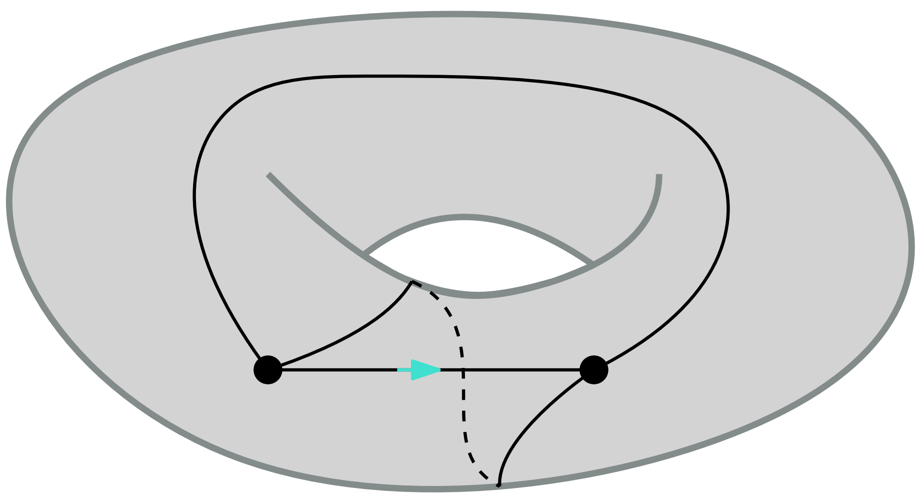
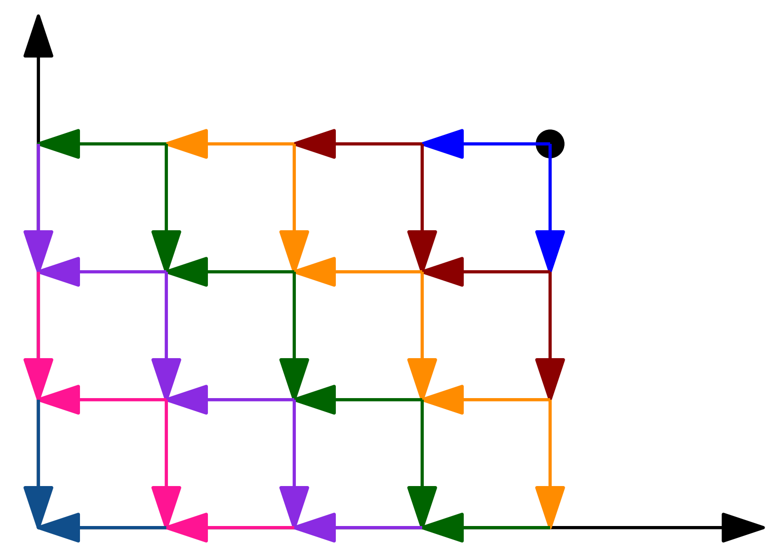
Counting with random walks

Baptiste Louf
with Andrew Elvey–Price, Wenjie Fang and Michael Wallner

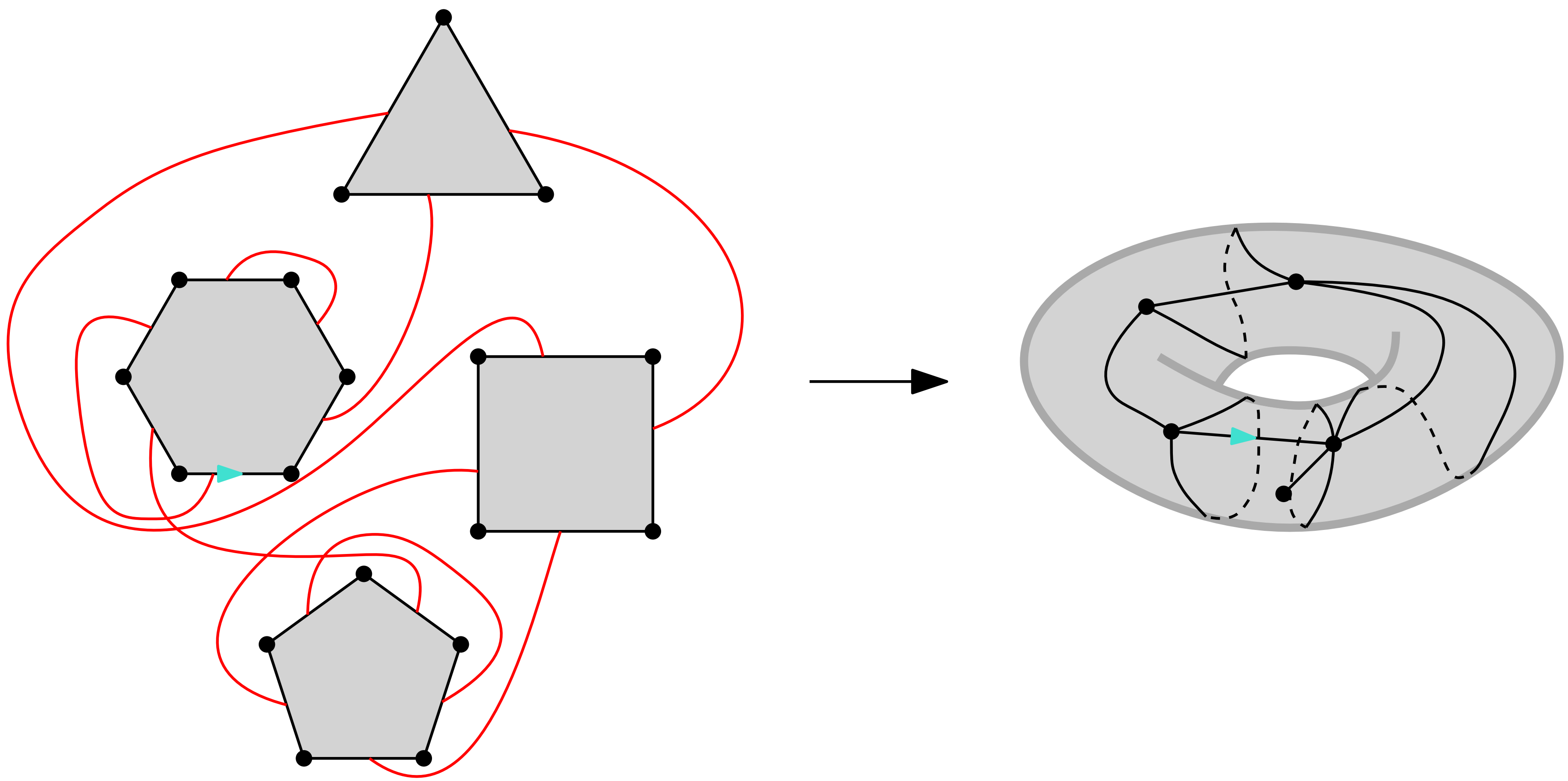




A bit of context:
combinatorial maps and enumerative combinatorics

Definition : maps

Map = **discrete surfaces**
i.e. gluing of polygons along their edges to create a (compact, connected, oriented) surface
Genus g of the map = genus of the surface = # of handles



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Exact answer for **planar maps** (i.e in genus 0)

Theorem [Tutte '60s]:
a_n = nb of planar maps with n edges

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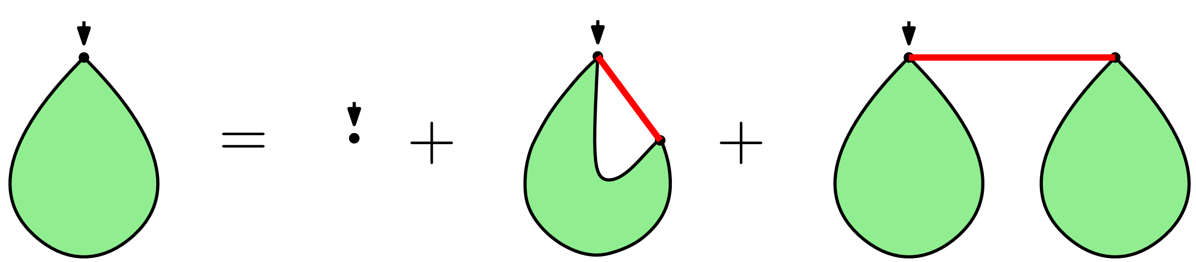
Method : Generating function and recursive decomposition

$$F(z) = \sum_{n \geq 0} a_n z^n$$



Counting maps . . . asymptotically

How about maps in positive genus ?

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Theorem [Lehman–Walsh '72+ Bender–Canfield '86] :
$a_n^{(g)}$ = nb of maps of genus g with n edges

$$a_n^{(g)} \sim C_g 12^n n^{5/2(g-1)}$$

as $n \to \infty$ for g fixed.

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What if n and g both go to ∞ ?

Bivariate asymptotics

If $n, g \to \infty$, we're dealing with **bivariate asymptotics**
$\to$ it is notoriously difficult !

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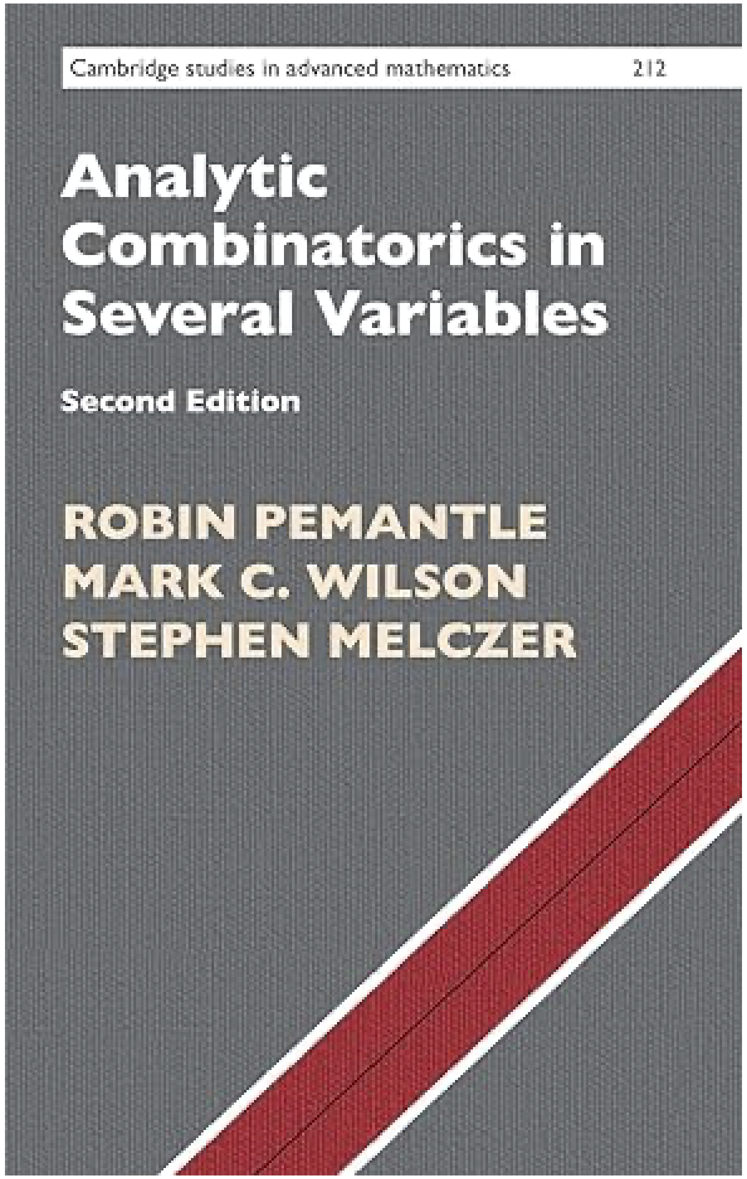

Recent progress to get asymptotics when the generating function is explicit and "simple"

(2nd edition, 2024, 550 pages !)

Bivariate asymptotics

If $n, g \to \infty$, we're dealing with **bivariate asymptotics**
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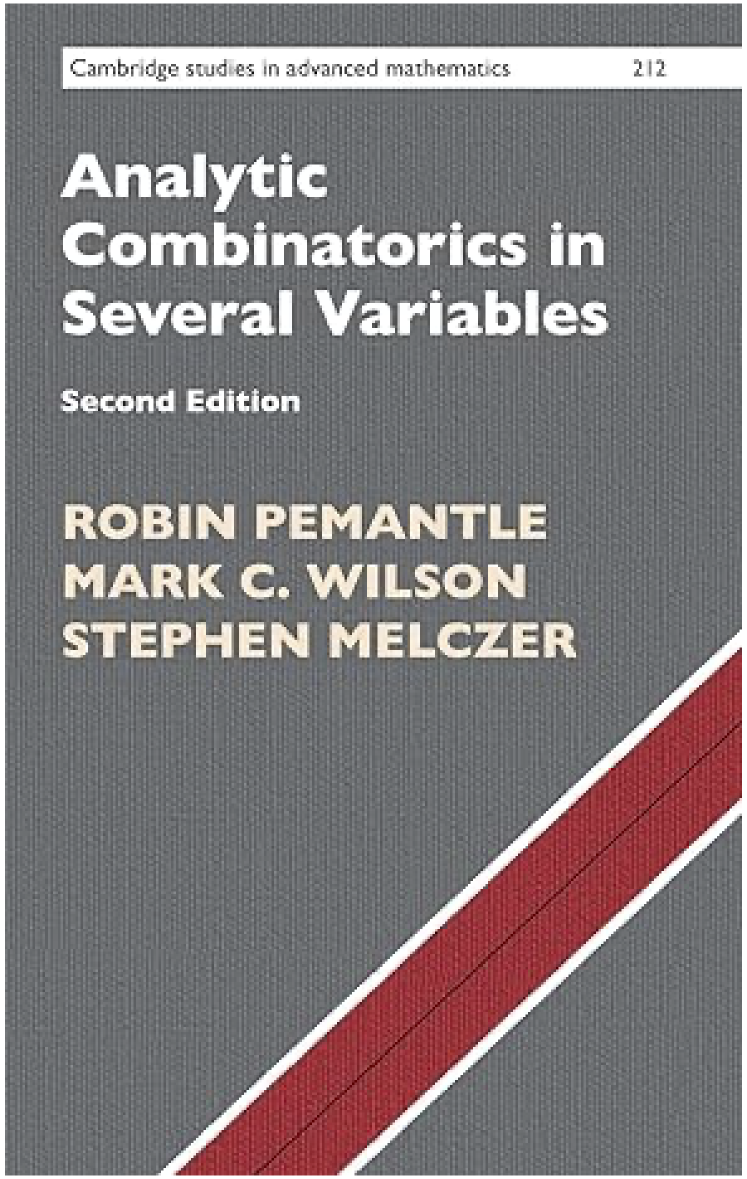

Recent progress to get asymptotics when the generating function is explicit and "simple"

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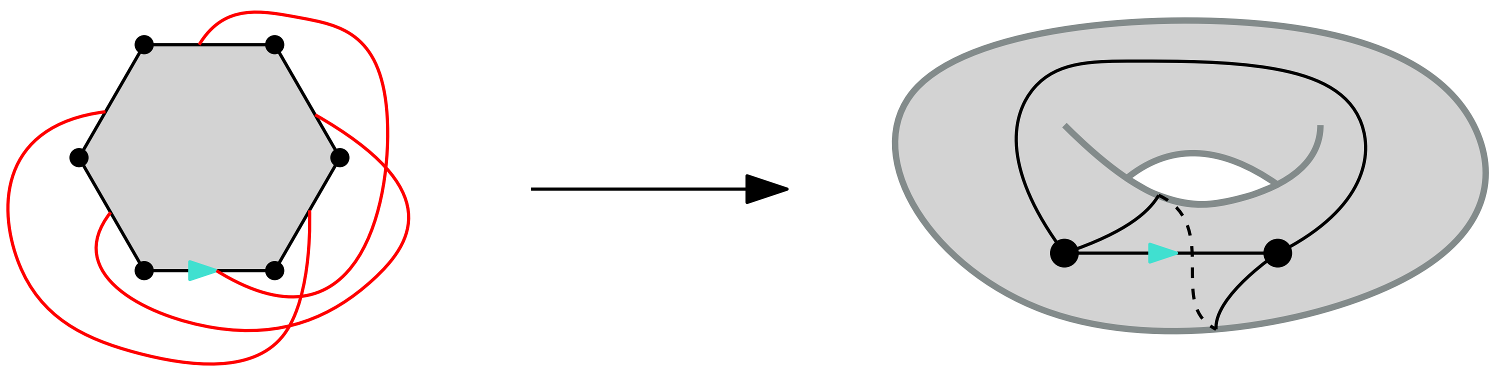
$\to$ maps do not fit in this case !

The problem: enumerating unicellular maps, asymptotically, bivariately

Unicellular maps

Simplest model of maps: maps with only one face/gluing of a single polygon



(unicellular map of genus 0 = tree !)

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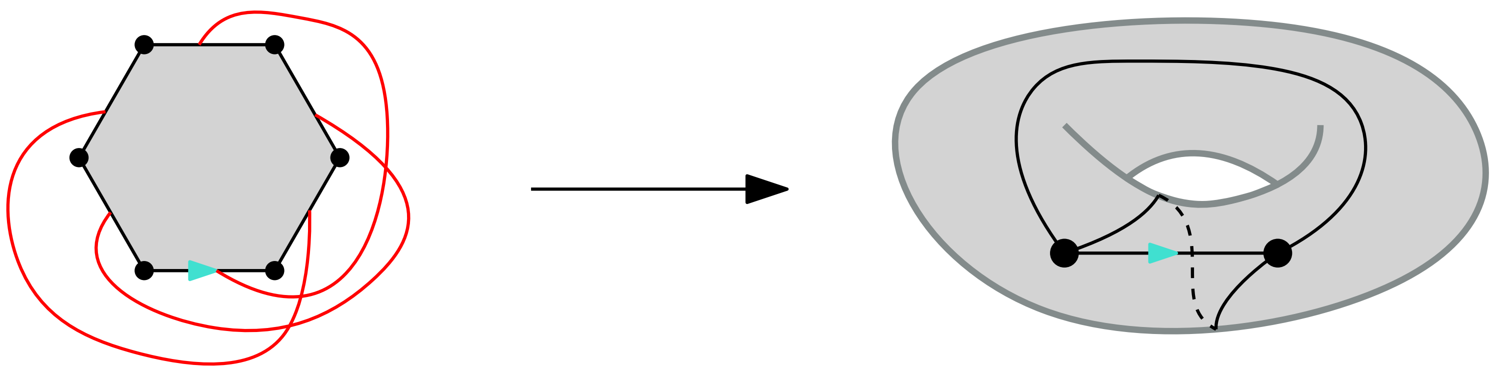

(unicellular map of genus 0 = tree !)

Let $E(n, g)$ be the number of unicellular maps with n edges and genus g
Goal: Study the asymptotics of $E(n, g)$ as $n, g \to \infty$!

Unicellular maps: what's known ?

Theorem [Harer–Zagier '86]: $E(0,0) = 1$, for $n \geq 1, n \geq 2g$, we have

$$(n+1)E(n,g) = 2(2n-1)E(n-1,g) + (n-1)(2n-3)(2n-1)E(n-2,g-1)$$

$$\implies 1 + 2xy + 2 \sum_{g \geq 0, n > 0} \frac{E(n,g)}{(2n-1)!!} y^{n+1} x^{n+1-2g} = \left(\frac{1+y}{1-y}\right)^x$$

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Asymptotic enumeration:

- for $\frac{g}{n} \to \theta \in (0, 1/2)$ [Angel–Chapuy–Curien–Ray '13]
- for $g = O(n^{1/3})$ [Curien–Kortchemski–Marzouk '23]

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Method : A bijection between unicellular maps and decorated trees [Chapuy–Féray–Fusy '12] (first case) core/kernel decomposition (second case)

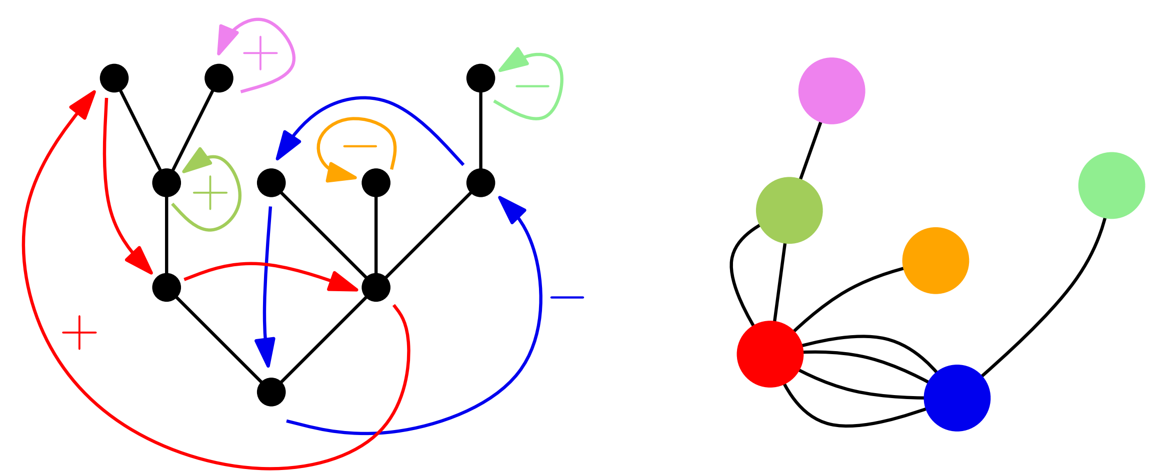

image : G. Chapuy

Univellular maps: full asymptotics

Our goal: Obtain asymptotics for $E(n,g)$ for all regimes of n,g using **only** the Harer-Zagier recurrence (we forget about the combinatorics !)

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Theorem: [Elvey-Price–Fang–L.–Wallner '2x]
As $n, g \to \infty$ with $n - 2g >> \log(n)$

$$E(n,g) \sim \frac{1}{2\sqrt{2}\pi} n^{2g-2} e^{nf(\frac{g}{n})} J\left(\frac{g}{n}\right),$$

with

$$\theta(\lambda) = \frac{1}{2} - \frac{\lambda \log\left(\frac{1+\sqrt{1-4\lambda}}{1-\sqrt{1-4\lambda}}\right)}{\sqrt{1-4\lambda}},$$

$$f(\theta) = -\theta \log\left(\frac{1-4\lambda}{4\lambda^2}\right) - 2\theta - \log(\lambda),$$

$$J(\theta) = \sqrt{\frac{2}{\lambda(\theta)(1-4\lambda(\theta)-2\theta+4\theta\lambda(\theta))}}.$$

Idea of proof 1: guess and check

Guess and check

$$\textbf{(HZ)} \quad (n+1)E(n,g) = 2(2n-1)E(n-1,g) + (n-1)(2n-3)(2n-1)E(n-2,g-1)$$

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Idea: if I have explicit formulas for number $\Omega(n,g)$ such that they satisfy $(\mathbf{HZ})$ and $\Omega(0,0) = E(0,0)$, then $E(n,g) = \Omega(n,g)$ always !

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Goal: find numbers $\Omega(n,g)$ such that:

$$\Omega(n,0) \sim E(n,0) \text{ as } n \to \infty$$

"asymptotic initial condition"

$$\begin{aligned}(n+1)\Omega(n,g) \approx\ & 2(2n-1)\Omega(n-1,g) \\ & + (n-1)(2n-3)(2n-1)\Omega(n-2,g-1)\end{aligned}$$

"asymptotic recurrence"

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Then hopefully

$$\Omega(n,g) \sim E(n,g)$$

Idea of proof 2: random walks

Rewriting the recurrence in terms of walks

Set $A(v,g) := E(v+2g,g)$ Harer–Zagier rewrites

$$A(\mathbf{v},\mathbf{g}) = \frac{2(2n-1)}{n+1}A(\mathbf{v}-\mathbf{1},\mathbf{g}) + \frac{(n-1)(2n-3)(2n-1)}{n+1}A(\mathbf{v},\mathbf{g}-\mathbf{1})$$

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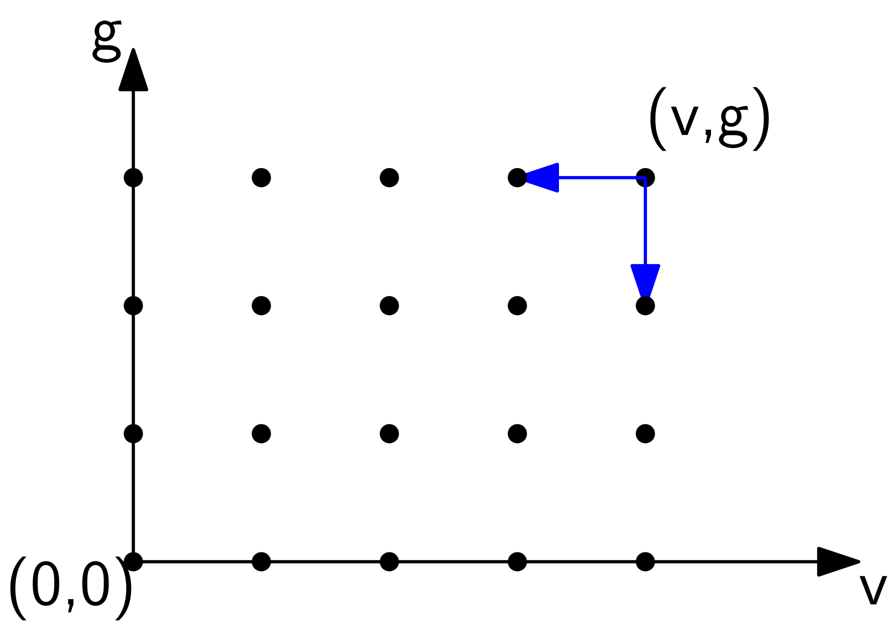

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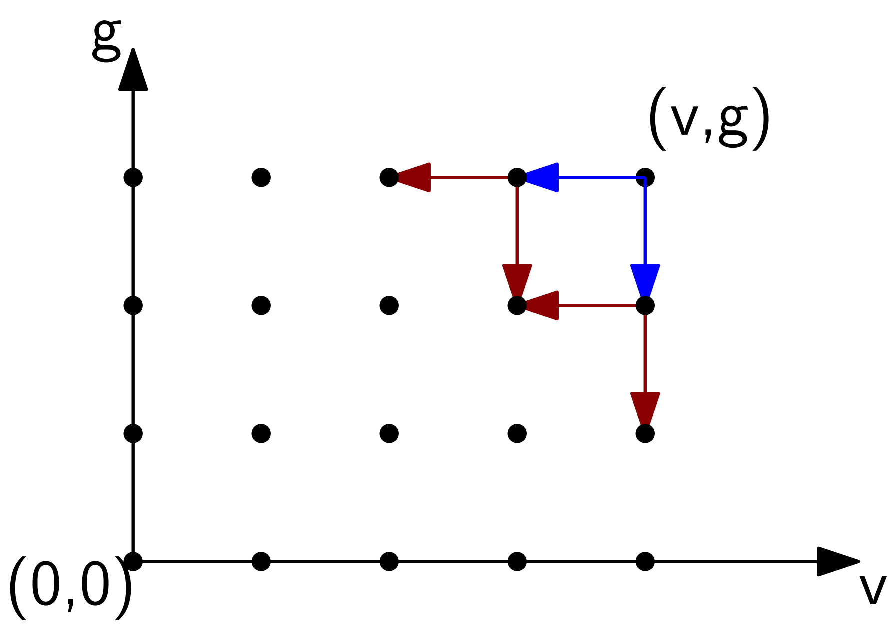

$$\begin{aligned}
&= \frac{2(2n-1)2(2n-3)}{(n+1)n}A(\mathbf{v-2},\mathbf{g}) \\
&+ \frac{2(2n-1)(n-2)(2n-5)(2n-3)}{(n+1)n}A(\mathbf{v-1},\mathbf{g-1}) \\
&+ \frac{(n-1)(2n-3)(2n-1)2(2n-5)}{(n+1)(n-1)}A(\mathbf{v-1},\mathbf{g-1}) \\
&+ \frac{(n-1)(2n-3)(2n-1)(n-3)(2n-7)(2n-5)}{(n+1)(n-1)}A(\mathbf{v},\mathbf{g-2})
\end{aligned}$$

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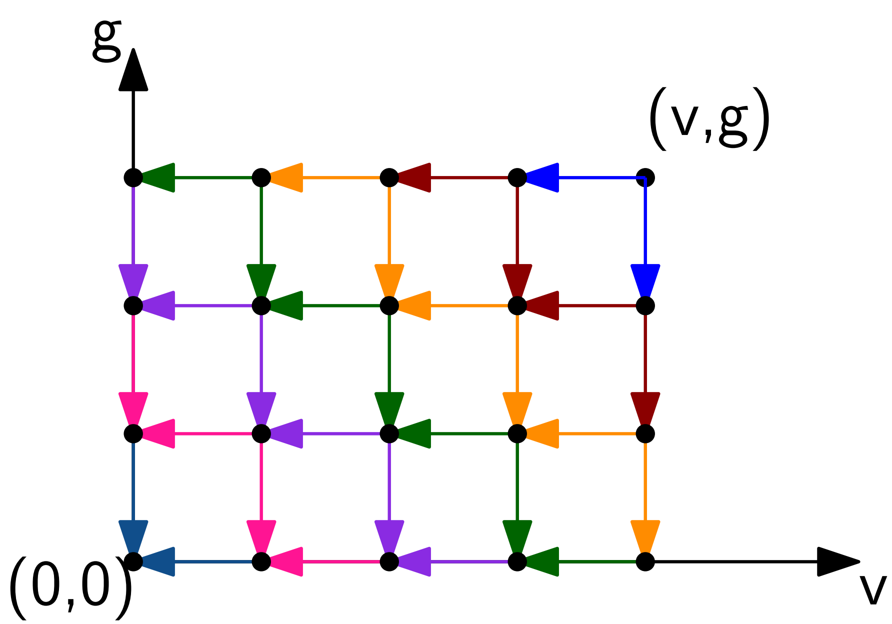

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$$= \sum_{\text{paths from } (v,g) \text{ to } (0,0)} \prod_{\text{steps of the paths}} weight(step)$$

(because $A(0,0) = 1$!)

Modelling by random walks: first ideas

Question: What are the paths that contribute to the counting ?
Behaviour of RW started from N_0, G_0, with weight steps:

$$\frac{2(2n-1)}{n+1}\frac{E(n-1,g)}{E(n,g)} \quad \text{and} \quad \frac{(n-1)(2n-3)(2n-1)}{n+1}\frac{E(n-2,g-1)}{E(n,g)}$$

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Approximation goal: Find $\Omega(n,g)$ such that

$$\frac{2(2n-1)}{n+1}\frac{\Omega(n-1,g)}{\Omega(n,g)} + \frac{(n-1)(2n-3)(2n-1)}{n+1}\frac{\Omega(n-2,g-1)}{\Omega(n,g)} \approx 1$$

Proof: more details

Defining $\Omega(n, g)$

Setup:

$$\Omega(n,g) := \frac{1}{2\sqrt{\pi}} \frac{\sqrt{g}(g/e)^g}{g!} n^{2g-2} e^{nf\left(\frac{g}{n}\right)} J\left(\frac{g}{n}\right) \frac{\sqrt{2\pi}(n-2g)^{n-2g+1}}{e^{(n-2g)}\Gamma(n-2g+3/2)},$$

$$\alpha(n,g) := \frac{2(2n-1)}{n+1} \frac{\Omega(n-1,g)}{\Omega(n,g)}$$

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Key property: for $n > 2g$ and $g > 0$:

$$\alpha(n,g) + \beta(n,g) := 1 + O\left(\frac{1}{n\log^2(n)}\right)$$

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← **summable !**

This means that our approximation by a random walk will be valid !

Defining the random walk

Setup: Start from $N_0, G_0 = n, g$, stop when $G_k = 0$ or $N_k = 2G_k$.
Stopping time $\tau = \tau(n, g)$

$$(N_{k+1}, G_{k+1}) = (N_k - 1, G_k) \quad \text{with proba } \frac{\alpha(N_k, G_k)}{\alpha(N_k, G_k) + \beta(N_k, G_k)}$$

$$(N_{k+1}, G_{k+1}) = (N_k - 2, G_k - 1) \quad \text{with proba } \frac{\beta(N_k, G_k)}{\alpha(N_k, G_k) + \beta(N_k, G_k)}$$

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Conserved quantity:

$$Q(n, g) := \frac{E(n, g)}{\Omega(n, g)}$$

(HZ) rewrites

$$Q(n, g) = \alpha(n, g)Q(n-1, g) + \beta(n, g)Q(n-2, g-1)$$

Hence

$$\mathbb{E}(Q(N_{k+1}, G_{k+1})) \approx \mathbb{E}(Q(N_k, G_k))$$

Typical behaviour and asymptotic result

Typical behaviour:

Propostion: As $n, g \to \infty$ with $n - 2g >> \log n$, with "very high probability":

$$G_\tau = 0 \quad \text{and} \quad N_\tau \to \infty$$

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Asymptotics as a corollary:

Since $Q(n, 0) \to 1$ as $n \to \infty$,

$$\mathbb{E}(Q(N_k, G_k) \sim 1$$

, but

$$\mathbb{E}(Q(N_k, G_k) \sim Q(N_0, G_0) = Q(n, g)$$

hence

$$E(n, g) \sim \Omega(n, g)$$

Typical behaviour: proof

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Proof: If time permits, see the board !

How to guess ?

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Plotting:

Fix $\theta \in (0, 1/2)$ and plot

$$\frac{E(\lfloor \theta g \rfloor, g-1)}{E(\lfloor \theta g \rfloor, g)}$$ it grows like g^2

$$\frac{E(\lfloor \theta g \rfloor - 1, g)}{E(\lfloor \theta g \rfloor, g)}$$ it converges but depends on θ

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"Guess and check":

$$E(n,g) \approx n^{2g} exp\left(nf(g/n)\right)$$

(HZ) gives a differential equation for f.

More details ? If time permits, see the board !

Conclusion

Recap: Approximating a linear bivariate recurrence by a random walk
Guess and check type of method
Nice feature: once the RW is defined, we only have to manipulate explicit formulas (a bit tedious though)

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Perspectives:

- linear $\rightarrow$ quadratic
- sticky walls $\rightarrow$ bouncy walls
- coefficients depend only on n $\rightarrow$ coefficients depending on n and g

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Nice feature: once the RW is defined, we only have to manipulate explicit formulas (a bit tedious though)

Robust method: Doesn't rely on the combinatorics of the model, in fact we can study other linear recurrences of the same kind !
$\rightarrow$ Open question: how large is the class of recurrences tackled by our method ?

Perspectives:

- linear $\rightarrow$ quadratic
- sticky walls $\rightarrow$ bouncy walls
- coefficients depend only on n $\rightarrow$ coefficients depending on n and g

Other works on recurrences and random walks:
[Aggarwal '18,'20, Elvey-Price–Fang–Wallner '19,'20, Chassaing–Flin '22,...]

Thank you !

Postdoc in Bordeaux with me starting Fall 2025
On “High genus geometry and/or asympotic enumeration”
To be announced soon