A view on height coupled trees

Meltem Ünel, October 2024, MathSTIC Days

Joint work(s) with B. Durhuus.

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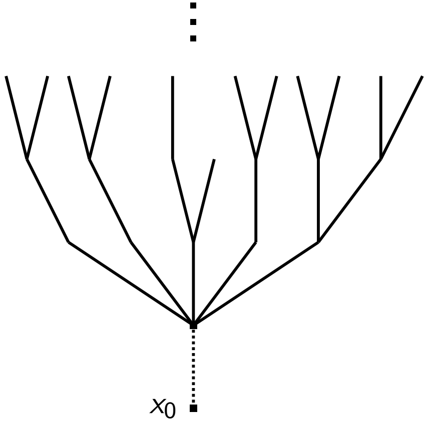
Metric space of rooted planar trees



- Degree of a vertex $j \in V(T)$: σ_j ,
- *Sectors* around j : $(S_{j,1}, \ldots, S_{j,\sigma_j})$.

- $\mathcal{T}_{\text{fin}} = \bigcup_{N=1}^{\infty} \mathcal{T}_N \quad , \quad \mathcal{T} = \mathcal{T}_{\text{fin}} \cup \mathcal{T}_\infty,$

- Subgraph $B_R(T)$ of T spanned by $V(B_R(T)) = \bigcup_{s=0}^{R} D_s(T)\,,$

- For $T, T' \in \mathcal{T}$

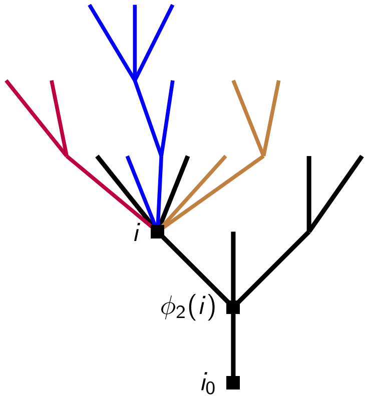
$$\text{dist}(T, T') = \inf\{\frac{1}{R} \mid R \in \mathbb{N},\ B_R(T) = B_R(T')\}\,.$$

- $(\mathcal{T}, \text{dist})$ is separable and complete.

- The ball of radius $\frac{1}{R}$ around $T_0 \in \mathcal{T}$

$$\mathcal{B}_{\frac{1}{R}}(T_0) = \{T \in \mathcal{T} \mid B_R(T) = B_R(T_0)\}\,.$$

- Infinite-type vertices span a subtree with no leaves: *BACKBONE* or *SPINE* of T.
- $\chi : \mathcal{T}_\infty \to \mathcal{T}_\infty$ is Borel measurable.
- $\mathcal{T}^s := \chi(\mathcal{T}_\infty)$.



A sequence ν_N converges weakly to ν on $\mathcal{T}$

$$\int_{\mathcal{T}} f d\nu_N \to \int_{\mathcal{T}} f d\nu$$

as $N \to \infty$ for all bounded continuous functions f on $\mathcal{T}$.

- Any ball in $\mathcal{T}$ is both open and closed,
- Two balls are either disjoint or one is contained in the other.

So, ν_N converges to ν if for any ball $B \in \mathcal{T}$

$$\nu_N(B) \to \nu(B) \text{ as } N \to \infty\,.$$

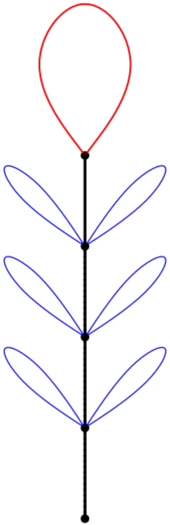
Local limit

Generic case: UIPT

$$\nu_N^{(0)}(T) = \frac{1}{|\mathcal{T}_N|} = \frac{1}{C_{N-1}}\,, \quad T \in \mathcal{T}_N\,.$$

Kesten tree:

$$\nu_N^{(0)} \xrightarrow[N\to\infty]{} \nu^{(0)}\,.$$



Local limit

Generic case: UIPT

$$\nu_N^{(0)}(T) = \frac{1}{|\mathcal{T}_N|} = \frac{1}{C_{N-1}}\ , \quad T \in \mathcal{T}_N\,.$$

Height dependent weights

$$\nu_N(T) = \frac{f(h(T))}{Z_N}\,,\ \ T \in \mathcal{T}_N$$

where

$$Z_N = \sum_{T \in \mathcal{T}_N} f(h(T))\,.$$

Generating Functions

- $X_m(g) = \sum_{h(T)\leq m} g^{|T|} = \sum_{N=1}^{\infty} A_{m,N} g^N$.

- Let $z = \sqrt{1-4g}$.

$$X_m(g) = 2g \frac{(1+z)^m - (1-z)^m}{(1+z)^{m+1} - (1-z)^{m+1}}\,, m \geq 0,$$
$$= \sqrt{g} \frac{U_{m-1}(\frac{1}{2\sqrt{g}})}{U_m(\frac{1}{2\sqrt{g}})}\,.$$

- Poles at $z_p = \pm i \tan\left(\frac{p}{m+1}\pi\right)$, $p = 0, 1, \ldots, \lfloor \frac{m}{2} \rfloor$ and residues can be calculated from which

$$A_{m,N} = 4^N \sum_{k=1}^{\lfloor \frac{m}{2} \rfloor} \frac{1}{m+1} \tan^2 \frac{\pi k}{m+1} \left(1 + \tan^2 \frac{\pi k}{m+1}\right)^{-N}, \quad N \geq 2\,,$$

Powerlike case: $f = h(T)^\alpha$

- $$\mathbb{W}_\alpha(g) = \sum_{N=1}^{\infty} \sum_{T \in \mathcal{T}_N} h(T)^\alpha g^{|T|} = \sum_{N=1}^{\infty} Z_N g^N$$
  $$= \sum_{m=1}^{\infty} m^\alpha (X_m - X_{m-1})\,.$$

- $$X_m - X_{m-1} = \frac{z^2}{\frac{1+z}{2}(\frac{1+z}{1-z})^m + \frac{1-z}{2}(\frac{1-z}{1+z})^m - 1}.$$

- Sequence of lemmas: study the power series of $X_m - X_{m-1}$.

Theorem (Durhuus, Ü.)

Let $a > 0$. Then in all three cases below, $\mathbb{W}_\alpha$ is analytic in the right half-plane $\mathbb{C}_+ = \{z \in \mathbb{C} \mid \operatorname{Re} z > 0\}$ and for z small in V_a, the following statements hold.

- *Assume the exponent α fulfills $-(2n+1) < \alpha < -(2n-1)$, $n \in \mathbb{N}_0$. Then there exists $\Delta > 0$ such that*

$$\mathbb{W}_\alpha(z) = \mathbb{W}_\alpha^{(n)}(z) + c_\alpha z^{1-\alpha} + O(|z|^{1-\alpha+\Delta}) \tag{1}$$

where

$$\mathbb{W}_\alpha^{(n)}(z) = \sum_{m=1}^{\infty} \frac{m^\alpha}{2m(m+1)} \big(1 + \sum_{k=1}^{n} c_{2k}^m (2mz)^{2k}\big)\,, \tag{2}$$

$$c_\alpha = \int_0^\infty \Big(\frac{t^\alpha}{\cosh(2t) - 1} - t^\alpha L_n(2t)\Big) dt\,,$$

and $L_n(t)$ is the Laurent polynomial of order $2(n-1)$ of $\frac{1}{\cosh t - 1}$.

Theorem (Cont.)

- *Assume* $\alpha > 1$. *Then there exists* $\Delta > 0$ *such that*

$$\mathbb{W}_\alpha(z) = c_\alpha z^{1-\alpha} + O(|z|^{1-\alpha+\Delta}) \tag{3}$$

where

$$c_\alpha = \int_0^\infty \frac{t^\alpha}{\cosh(2t) - 1} dt\,.$$

- *Assume* $\alpha = -(2n-1)$, $n \in \mathbb{N}_0$. *Then there exists a polynomial* $\mathbb{P}_n(z)$ *of degree* $2n$ *and a constant* $\Delta > 0$ *such that*

$$\mathbb{W}_\alpha(z) = \mathbb{P}_n(z) + d_n z^{2n} \ln z + O(|z|^{2n+\Delta}) \tag{4}$$

where, denoting the Taylor coefficients of $2t^2(\cosh t - 1)^{-1}$ *by* A_n,

$$d_n = -2^{2n-1} A_n\,.$$

Proposition

For fixed $\alpha \in \mathbb{R}$ it holds for large N that

$$[g^N]\mathbb{W}_\alpha = Z_N = K_\alpha \, N^{\frac{\alpha-3}{2}} 4^N (1 + o(1)) \,, \tag{5}$$

where the constant K_α is given by

$$K_\alpha = \begin{cases} \frac{c_\alpha}{\Gamma(\frac{\alpha-1}{2})} \,, & \text{if } \alpha > 1 \text{ or if } -(2n+1) < \alpha < -(2n-1),\ n \in \mathbb{N}_0 \,, \\ 4^{n-1}|A_n| n! \,, & \text{if } \alpha = -(2n-1) \,,\ n \in \mathbb{N}_0 \,. \end{cases}$$

In the particular case $\alpha = 0$

$$Z_N = C_{N-1} := \frac{(2(N-1))!}{N!(N-1)!} = \frac{1}{\sqrt{\pi}} N^{-\frac{3}{2}} 4^{N-1} \left(1 + O(N^{-1})\right) .$$

Local limit

Proposition

Let $T_0 \in \mathcal{T}_{\text{fin}}$ have height r and assume it has R vertices at height r. For every $\epsilon > 0$ there exists $M_0 \in \mathbb{N}$ s.t. for all $M > M_0$

$$\nu_N(\mathcal{B}_{\frac{1}{r}}(T_0)) \geq (1-\epsilon)R \cdot 4^{R-|T_0|}\Big(\sum_{K=1}^{M} C_{K-1}4^{-K}\Big)^{R-1}(1+o(1))\,.$$

Theorem (Durhuus, Ü.)

The sequence (μ_N) converges weakly to UIPT which is characterised by

$$\nu(\mathcal{B}_{\frac{1}{r}}(T_0)) = R \cdot 2^{R+1}4^{-|T_0|}\,, \tag{6}$$

for any tree $T_0 \in \mathcal{T}_{\text{fin}}$ of height r, where R denotes the number of vertices at height r.

$f = e^{-\mu h(T)}$

Generic case: UIPT

$$\nu_N^{(0)}(T) = \frac{1}{C_{N-1}}\,, \quad T \in \mathcal{T}_N\,.$$

Exponential height dependent weights

$$\nu_N^{(\mu)}(T) = \frac{e^{-\mu h(T)}}{Z_N^{(\mu)}}\,, \; T \in \mathcal{T}_N\,.$$

$$Z_N^{(\mu)} = \sum_{T \in \mathcal{T}_N} f(h(T)) = [g^N] \sum_{m=1}^{\infty} f(h(T))(X_m - X_{m-1})$$
$$= \sum_{m=1}^{\infty} e^{-\mu m}(A_{m,N} - A_{m-1,N})\,.$$

$\mu > 0$: Coefficient Asymptotics

Proposition

For each $\mu > 0$ it holds for any $\delta \in]0, \frac{1}{6}[$ that

$$Z_N^{(\mu)} = (e^{\mu} - 1)\sqrt{\frac{\pi}{B}}\frac{\mu}{2}e^{-AN^{\frac{1}{3}}}N^{-\frac{5}{6}}4^N\Big(1 + O(N^{-\delta})\Big),$$

for N large, where

$$A = 3\Big(\frac{\pi\mu}{2}\Big)^{\frac{2}{3}} \quad \textit{and} \quad B = 3\Big(\frac{\mu^2}{4\pi}\Big)^{\frac{2}{3}}.$$

Same results via Dyck paths: *Guttmann A.J. , Analysis of series expansions for non-algebraic singularities, 2015*.

Local limit

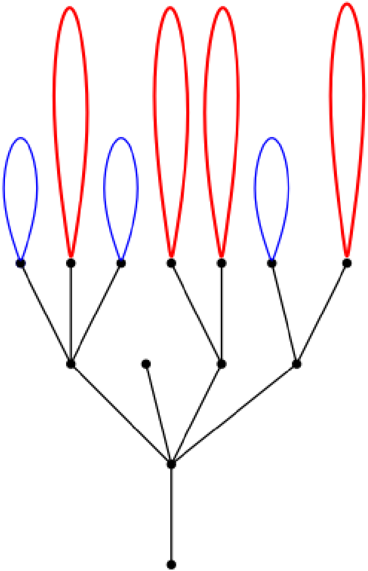
Theorem (Durhuus, M.Ü.)

For each $\mu > 0$ the sequence $(\nu_N^{(\mu)})$ is weakly convergent to a Borel probability measure $\nu^{(\mu)}$ on $\mathcal{T}$ characterized by

$$\nu^{(\mu)}(\mathcal{B}_{\frac{1}{r}}(T_0)) = \frac{e^{-\mu(r-1)}}{4^{|T_0|}} 2^{K+1} \sum_{R=1}^{K} \binom{K}{R} \frac{\mu^{R-1}}{(R-1)!},$$

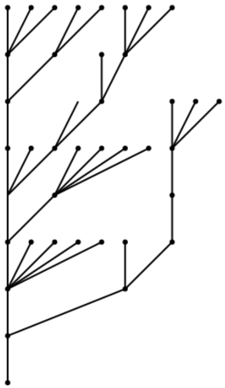
for any tree $T_0 \in \mathcal{T}_{\mathrm{fin}}$ of height $r \geq 1$, where $K = |D_r(T_0)|$, and which equals the infinite Poisson tree $\nu_{Poi}^{(\mu)}$ with parameter μ.

Proof by Portmenteau theorem .



One-sided trees

- We say that T is a *one-sided tree* if $|\omega_T| = h(T)$, where ω_T is the left-most path in T,
- $\mathcal{T}_N$, $\mathcal{T}_\infty$, $\mathcal{T}_{\mathrm{fin}}$, $\mathcal{T} \to \Omega_N$, Ω_∞, Ω_{fin}, Ω,
- $\Omega^{(m)}$, $\Omega_N^{(m)}$: N for size, m for height.



Generating functions

$$Y_m(g) = \sum_{T \in \Omega^{(m)}} g^{|T|} := \sum_{N=1}^{\infty} B_{m,N} g^N = X_m(g) Y_{m-1}(g)\,, \quad m \geq 2\,.$$

$$Y_1(g) = X_1(g) = g \implies Y_m(g) = \prod_{k=1}^{m} X_k(g) = \frac{g^{\frac{m}{2}}}{U_m(\frac{1}{2\sqrt{g}})}\,.$$

$$B_{m,N} = \frac{2^{-(m-1)}}{m+1} \sum_{k=1}^{\lfloor \frac{m}{2} \rfloor} (-1)^{k+1} \tan^2 \frac{\pi k}{m+1} \left(1 + \tan^2 \frac{\pi k}{m+1}\right)^{-N + \frac{m-1}{2}} \cdot 4^N$$

Compare with

$$A_{m,N} = \frac{1}{m+1} \sum_{k=1}^{\lfloor \frac{m}{2} \rfloor} \tan^2 \frac{\pi k}{m+1} \left(1 + \tan^2 \frac{\pi k}{m+1}\right)^{-N} \cdot 4^N$$

The case $\mu = \mu_0 = -\ln 2$

We analyse it with similar tools as in the power-like case.

Proposition

There exists a constant $c_0 \in \mathbb{R}$ such that for any given $a > 0$ we have

$$W^{(\mu_0)}(g) = -\frac{1}{2}\ln(1-4g) + c_0 + O(|1-4g|^{\frac{1}{5}})$$

for all complex g close to $\frac{1}{4}$ satisfying $\sqrt{1-4g} \in V_a$.

Corollary

For N large, it holds that

$$W_N^{(\mu_0)} = \frac{1}{2N} \cdot 4^N \big(1 + o(1)\big) .$$

The case $\mu = \mu_0 = -\ln 2$

The same as before, call the finite size height-weighted measures by $\tau_N^{(\mu)}$.

Theorem (Durhuus, M.Ü.)

The measures $\tau_N^{(\mu_0)}$ *converge weakly to the probability measure* $\tau^{(\mu_0)}$ *on* Ω_∞ *defined by*

$$\tau^{(\mu_0)}(\mathcal{B}_{\frac{1}{r}}(T_0)) = 4^{-|T_0|}2^{K+r}\,,$$

for any tree $T_0 \in \Omega$ *of height* r *and with* K *vertices at height* r.

This measure equals the BGW measure corresponding to the BGW process with two types of individuals defined by the offspring probabilities $p(n) = 2^{-(n+1)}, n = 0, 1, 2, \ldots,$ *where* n *is the number of normal individuals.*

$\mu > -\ln 2$ case

Lemma

Assume $\mu > \mu_0$ *and that* $T_0 \in \Omega^{(r)}$ *and set* $K = |D_r(T_0)|$. *Given* $0 < \epsilon < \frac{1}{K}$ *and* $M \in \mathbb{N}$, *it holds for any* $\delta \in]0, \frac{1}{6}[$ *that*

$$\tau_N^{(\mu)}(\mathcal{B}_{\frac{1}{r}}(T_0)) \geq \frac{e^{-\mu(r-1)}}{4^{|T_0|-K}} \sum_{R=1}^{K} \binom{K-1}{R-1} \frac{1}{(R-1)!} \Big(\frac{\mu-\mu_0}{2}\Big)^{R-1} \cdot \Big(\sum_{S=1}^{M} C_{S-1} 4^{-S}\Big)^{K-R} (1-\epsilon K)^K (1 + O(N^{-\delta}))$$

for N *large.*

$\mu > -\ln 2$ case

$$\Xi(T_0) := \frac{e^{-\mu(r-1)}}{4^{|T_0|-K}} \sum_{R=1}^{K} \binom{K-1}{R-1} \frac{1}{(R-1)!} \left(\frac{\mu + \ln 2}{2}\right)^{R-1} 2^{R-K} .$$

Lemma

For all $r \geq 1$ it holds that

$$\sum_{T_0 \in \Omega^{(r)}} \Xi(T_0) = 1 .$$

From this follows the tightness of $\tau_N^{(\mu)}$.

Local limit

Theorem (Durhuus, M.Ü.)

For $\mu > \mu_0$, the sequence $(\tau_N^{(\mu)})$ is weakly convergent to a Borel probability measure $\tau^{(\mu)}$ on $\mathcal{T}$ characterized by

$$\tau^{(\mu)}(\mathcal{B}_{\frac{1}{r}}(T_0)) = \Xi(T_0) = \frac{e^{-\mu(r-1)}}{4^{|T_0|}}\, 2^{K+1} \sum_{R=1}^{K} \binom{K-1}{R-1} \frac{(\mu-\mu_0)^{R-1}}{(R-1)!},$$

for any tree $T_0 \in \Omega^{(r)}$, where $K = |D_r(T_0)|$ and $r \geq 1$. Moreover, the measure is concentrated on Ω_∞ and, in particular, $\tau^{(\mu)}(\mathcal{B}_{\frac{1}{r}}(T_0))$ vanishes unless T_0 is one-sided.

Use tightness or Portemanteau theorem for the proof.

Properties of $\tau^{(\mu)}$

- The (μ-scaled) simplex:

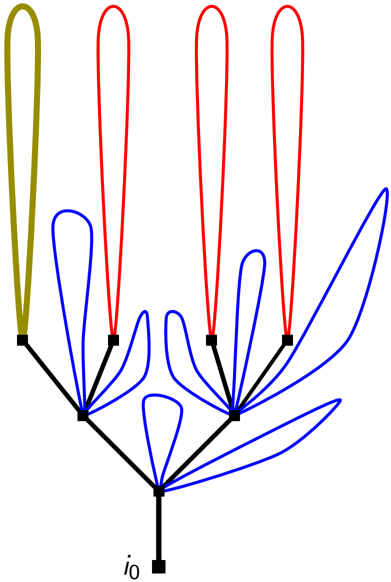
$$\mu\Delta_n := \{(\mu_1, \ldots, \mu_n) \mid \mu_1 + \cdots + \mu_n = \mu\,,\ \mu_1, \ldots, \mu_n > 0\}\,.$$

Proposition

Assume $\mu > \mu_0$ and that $T_0 \in \Omega^{(r)}$ and set $K = |D_r(T_0)|$. It holds for arbitrary $T_1, \ldots, T_K \in \mathcal{T}_{fin}$, with $h_i = h(T_i)$, that

$$\tau^{(\mu)}\,|_{\mathcal{B}_{\frac{1}{r}}(T_0)}\,(\mathcal{B}_{\frac{1}{h_1}}(T_1) \times \cdots \times \mathcal{B}_{\frac{1}{h_K}}(T_K)) = \sum_{\substack{D \subseteq D_r(T_0) \\ w_r \in D}} \frac{e^{-\mu(r-1)}}{4^{|T_0|}} 2^{K+1} \frac{(\mu-\mu_0)^{|D|-1}}{(|D|-1)!}$$

$$\cdot \Big[\int_{(\mu-\mu_0)\cdot\Delta_{|D|}} d\omega_{|D|}\, \tau^{(\mu_1+\mu_0)}(\mathcal{B}_{\frac{1}{h_1}}(T_1)) \prod_{i \in D\setminus\{w_r\}} \nu^{(\mu_i)}(\mathcal{B}_{\frac{1}{h_i}}(T_i)) \prod_{j \notin D} \rho(\mathcal{B}_{\frac{1}{h_j}}(T_j)) \Big]\,.$$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At the moment...

We ask the question, **what happens when μ is not constant but a function of N,** with *L. Addario-Berry, B. Corsini, N. Maîtra*.

Theorem (Addario-Berry, Corsini, Maîtra, Ü., +24)

For all $1/\sqrt{N} \ll \mu(n) \ll 1$, the local limit is the Uniform Infinite Planar Tree.

What about when $\mu \gg 1$?

Almost brownian regime: scaling limit

Theorem (Addario-Berry, Corsini, Maîtra, Ü., +24)

Assume $\lim_{n\to\infty} \mu_n n^{1/2} = \alpha \in \mathbb{R}$. *Let* $\mathbf{T}_n$ *be a random tree sampled from the height-weighted measure, and let* d_n *denote the graph distance on* $\mathbf{t}_n$. *Then,*

$$(\mathbf{T}_n, \frac{d_n}{\sqrt{2n}}) \xrightarrow{d} (\mathcal{T}_\alpha, d_\alpha),$$

in the GH sense. Letting the law of the excursion of $\mathcal{T}_\alpha$ *seen as a random variable on* $C[0,1]$ *be* ρ_α *and the law of a standard Brownian excursion* $(B(t) : t \in [0,1])$ *by* ρ_0, *then the Radon-Nikodym derivative satisfies*

$$\frac{d\rho_\alpha}{d\rho_0}(f) = e^{-\alpha||f||}.$$

- Rewrite $Z_N^{(\mu)} = (1 - e^{-\mu}) W_N + C_{N-1} e^{-\mu(N+1)}$ where

$$W_N = \sum_{m=1}^{N} e^{-\mu m} A_{m,N} \,,$$

- Consider the contribution $\tilde{W}_N$ to W_N obtained by the retaining only the first term in $A_{m,N}$

$$\tilde{W}_N := 4^N \sum_{m=2}^{N} \frac{e^{\mu}}{m+1} \tan^2 \frac{\pi}{m+1} \, e^{-f_N(m+1)} \,,$$

where $f_N(t) = \mu t + N \ln \left(1 + \tan^2 \frac{\pi}{t}\right), \quad t > 2 \,,$

- The unique minimum of $f_N(t)$ at $t_0 = \left(\frac{2\pi^2 N}{\mu}\right)^{\frac{1}{3}} + O(N^{-\frac{1}{3}}) \,.$

- Saddle point approximation: dominant contribution comes from the range $t_0 - N^{\frac{1}{3}-\delta} < m+1 < t_0 + N^{\frac{1}{3}-\delta}$ where $\delta \in]\frac{1}{9}, \frac{1}{6}[$,

- Approximate the sum by an (Gaussian) integral to get

$$\tilde{W}_N = e^{\mu}\sqrt{\frac{\pi}{B}}\frac{\mu}{2}e^{-AN^{\frac{1}{3}}}N^{-\frac{5}{6}}4^N\left(1 + O(N^{\frac{1}{3}-3\delta})\right),$$

- Show that the contribution to W_N from the left out terms is subdominant.

Lemma

Assume $\mu > 0$ and that $T_0 \in \mathcal{T}_{\mathrm{fin}}$ has height r, and set $K = |D_r(T_0)|$. Given $0 < \epsilon < \frac{1}{K}$ and $M \in \mathbb{N}$, it holds for any $\delta \in]0, \frac{1}{6}[$ and for N large that

$$\nu_N^{(\mu)}(\mathcal{B}_{\frac{1}{r}}(T_0)) \;\geq\; \frac{e^{-\mu(r-1)}}{4^{|T_0|-K}} \sum_{R=1}^{K} \binom{K}{R} \frac{1}{(R-1)!} \Big(\frac{\mu}{2}\Big)^{R-1} \Big(\sum_{S=1}^{M} C_{S-1} 4^{-S}\Big)^{K-R} (1-\epsilon K)^K (1 + O(N^{-\delta}))\,.$$

Take $N \to \infty$ then $\epsilon \to 0$, $M \to \infty$ limit of the RHS, one gets

$$\frac{e^{-\mu(r-1)}}{4^{|T_0|}} 2^{K+1} \sum_{R=1}^{K} \binom{K}{R} \frac{\mu^{R-1}}{(R-1)!}\,.$$

- $D_m(z) = \frac{1+z}{2}(\frac{1+z}{1-z})^m + \frac{1-z}{2}(\frac{1-z}{1+z})^m - 1$,

- In the right half plane, $\frac{|1+z|}{|1-z|} > 1$,

- For fixed such z, $\frac{z^2}{D_m}$ decays exponentially and $\mathbb{W}_\alpha = \sum m^\alpha \frac{z^2}{D_m}$ converges,

- Sequence of lemmas: study the power series of $\frac{z^2}{D_m(z)}$ **in the vedge**,

$$z^{1-\alpha} \sum_{m\geq 1} \frac{(zm)^\alpha z}{2z^2 m(m+1)} \left(1 + \sum_{k\geq 1} c_{2k}^m (2mz)^{2k}\right) \leftrightarrow \int \frac{t \cdot dt}{\cosh(2t) - 1}$$

$$\to z^{1-\alpha} \int \frac{(tz)^\alpha z \cdot dt}{\cosh(2tz) - 1} \to z^{1-\alpha} \int \frac{(tz)^\alpha z \cdot dt}{2z^2 t^2} \left(1 + \sum_{k\geq 1} A_k (2zt)^{2k}\right)$$

The case $\mu < 0$

Set $k := e^{-\mu} > 1$, $c(g) := \frac{g}{(1-X(g))^2}$ and $g_c(k) := \frac{k}{(k+1)^2}$.

Theorem [Durhuus, M.Ü.] : For fixed $k > 1$, $\exists\ b > g_c(k)$ s.t. $Z^{(\mu)}(g)$ is analytic in

$$\{g \in \mathbb{C} \mid |g| < b, g \neq g_c(k)\}\,,$$

and has a simple pole at $g_c(k)$.

Sketch of proof

- Determine the rate of convergence of X_m to X,
- Rewrite $k^m(X_{m+1}(g) - X_m(g)) = \frac{k(gf(g))^2}{1-X(g)}\big(kc(g)\big)^{m-1} + h_m(g)$, were $f(g)$ and h_m are analytic on $\mathbb{D}$,
- $|h_m(g)| \leq \mathrm{cst} \cdot \big(kc(|g|)^2\big)^m \quad , \quad |g| \leq b$ for any fixed $b < \frac{1}{4}$,
- $Z^{(\mu)}(g) = \frac{k(gf(g))^2}{\big(1-X(g)\big)\big(1-kc(g)\big)} + h(g)$ is analytic for $|g| < b$ except at $g_c(k)$.

Corollary : There exists $d > 0$ s.t.

$$Z_N^{(\mu)} = r(g_c(k))^{-(N+1)}\big(1 + O(e^{-dN})\big)$$

for N large, where r is the residue of $-Z^{(\mu)}(g)$ at $g = g_c(k)$.

Lemma : Let $k > 1$ and let $T_0 \in \mathcal{T}_{\text{fin}}$ have height r with $|D_r(T_0)| = K$. For each $M \in \mathbb{N}$, $\exists d > 0$ s.t.

$$\nu_N^{(\mu)}(\mathcal{B}_{\frac{1}{r}}(T_0)) \geq K \cdot k^{r-1} g_c(k)^{|T_0|-K} \Big(\sum_{S=1}^{M} C_{S-1} g_c(k)^S\Big)^{K-1} \big(1 + O(e^{-dN})\big)$$

$$\to K \cdot k^{r-1} \cdot g_c(k)^{|T_0|-K} X(g_c(k))^{K-1}.$$

- BGW tree corresponding to the branching process with offspring probabilities $p(n)$, $\sum_{n=0}^{\infty} p(n) = 1$, characterized by
$$\lambda(\mathcal{B}_{\frac{1}{r}}(T)) = \prod_{v \in \cup_{s=1}^{r-1} D_s(T)} p(\sigma(v) - 1)\,,$$

- Subcritical/critical $m := \sum_{n=0}^{\infty} np(n) < 1 \;/\; = 1\,,$

- Conditioning on height: local limit $\hat{\lambda}$ is supported on single spine trees with 2 types of individuals [Kesten, '86],

- Special individual : $p^*(n) = \frac{np(n)}{m}$, Normal individual : $p(n)$.

See [Athreya,Ney '72] or [Abraham,Delmas '15] for more details!

Local limit: $\mu < 0$

Theorem (Durhuus, M.Ü.)

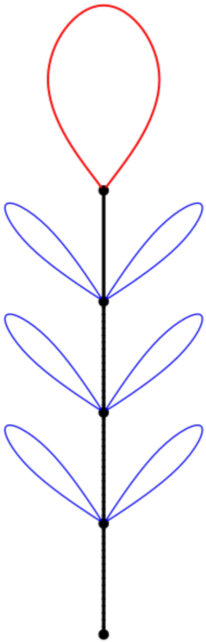
The sequence of measures $\nu_N^{(\mu)}$, $\mu < 0$ *converges weakly to the Kesten tree corresponding to the subcritical BGW tree with offspring probabilities*

$$p(n) = X(g_c(k))^n(1 - X(g_c(k))) \quad , \quad n = 0, 1, 2, \ldots .$$

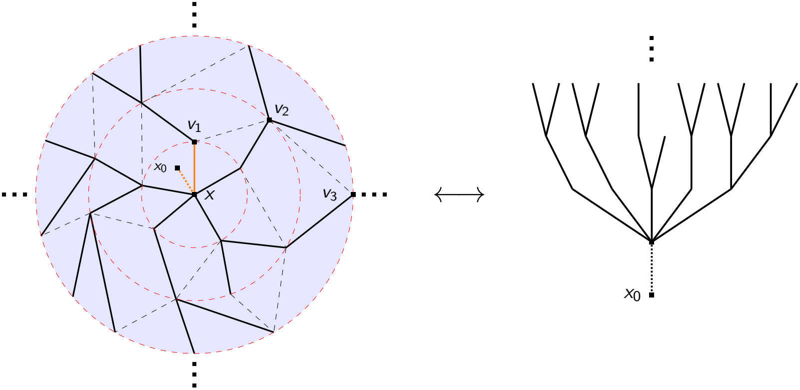
Corollary

$\mathbb{E}_\mu(|B_r|) = \frac{1+m}{1-m} \cdot r + O(1)$, *hence*

$$d_h = \lim_{r \to \infty} \frac{\ln \mathbb{E}_\mu(|B_r|)}{\ln r} = 1 \, .$$



Tree correspondences



Bijective correspondence, $\psi \ : \ \mathcal{C}_m \to \mathcal{T}_m$.

Complement: Coefficient Asymptotics $\mu > 0$

- Rewrite $Z_N = (1 - e^{-\mu}) W_N + C_{N-1} e^{-\mu(N+1)}$, where

$$W_N = \sum_{m=1}^{N} e^{-\mu m} A_{m,N} \,.$$

- Consider the term coming from the first term defining $A_{m,N}$, corresponding to the pole closest to 0.

$$\tilde{W}_N := 4^N \sum_{m=2}^{N} \frac{e^{\mu}}{m+1} \tan^2 \frac{\pi}{m+1} \; e^{-f_N(m+1)} \,,$$

where

$$f_N(t) = \mu t + N \ln\left(1 + \tan^2 \frac{\pi}{t}\right), \quad t > 2 \,.$$

- Estimate the unique minimum of $f_N(t)$: $\quad t_0 = \left(\frac{2\pi^2 N}{\mu}\right)^{\frac{1}{3}} + O(N^{-\frac{1}{3}})$.

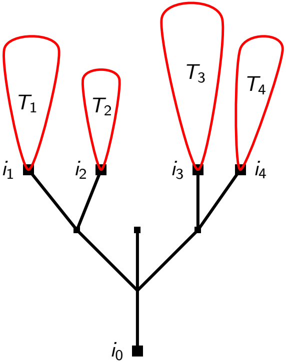
- Estimate in the regions $|m + 1 - t_0| > N^{\frac{1}{3}-\delta}$ and $|m + 1 - t_0| < N^{\frac{1}{3}-\delta}$ separately.
- The former is of lower order (compared to the stated one).
- In the latter, approximate it with a (restricted) Gaussian integral that approaches $\sqrt{\frac{\pi}{B}}$. This gives the stated quantity.
- Show that sum over the other "pole terms" is of lower order.

Local limit

Theorem [Durhuus,M.U.]: For each $\mu < 0$, $(\nu_N^{(\mu)})$ is weakly convergent to a Borel probability measure $\nu^{(\mu)}$ on $\mathcal{T}$, characterized by

$$\nu^{(\mu)}(\mathcal{B}_{\frac{1}{r}}(T_0)) = \Lambda(T_0) = K \cdot k^{r-1} \cdot g_c(k)^{|T_0|-K} X(g_c(k))^{K-1},$$

for any tree $T_0 \in \mathcal{T}_{\text{fin}}$ of height r, where $K = |D_r(T_0)|$.



- BGW tree corresponding to the branching process with offspring probabilities $p(n)$, $\sum_{n=0}^{\infty} p(n) = 1$, characterized by $\lambda(\mathcal{B}_{\frac{1}{r}}(T)) = \prod_{v \in \cup_{s=1}^{r-1} D_s(T)} p(\sigma(v) - 1)\,,$
- Subcritical/critical $m := \sum_{n=0}^{\infty} np(n) < 1 \ / \ = 1\,,$
- Conditioning on height: local limit is supported on single spine trees with 2 types of individuals [Kesten, '86],
- Branches grafted onto the vertices spanning the spine, to the left and right, are independent with identical distribution equal to the subcritical/critical BGW tree with offspring prob. p.

See [Athreya,Ney '72] or [Abraham,Delmas '15] for more details!

Properties of the limit: $\mu < 0$

Proposition (3.8): The measure $\nu^{(\mu)}$, $\mu < 0$, equals *Kesten tree* corresponding to the subcritical BGW tree with offspring probabilities

$$p(n) = X(g_c(k))^n(1 - X(g_c(k))) \quad , \quad n = 0, 1, 2, \ldots .$$

Corollary (3.9): $\mathbb{E}_\mu(|B_r|) = \frac{1+m}{1-m} \cdot r + O(1)$, hence

$$d_h = \lim_{r \to \infty} \frac{\ln \mathbb{E}_\mu(|B_r|)}{\ln r} = 1 \, .$$

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